1. Please write your 1- or 2-digit exam number on this cover sheet and on all problem sheets (even problems that you do not wish to be graded).

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3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem $p$ on either side of the page for problem $q$ if $p \neq q$.

4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

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Part A Subtotal: _____ Part B Subtotal: _____ Grand Total: _____
Problem 1A.  

(a) Prove that if $s > 1$ then $\sum_{n>0} n^{-s} = \prod_p 1/(1 - p^{-s})$, where the product is over all primes $p$.

(b) Prove that the sum $\sum_p 1/p$ over all primes $p$ diverges.

Solution:
Let $x: [a, b] \to \mathbb{R}$ and $f: [a, b] \to \mathbb{R}$ be non-negative continuous functions satisfying

$$x^2(t) \leq 1 + \int_a^t f(s)x(s)ds$$

for $a \leq t \leq b$. Show that

$$x(t) \leq 1 + \frac{1}{2} \int_a^t f(s)ds$$

for $a \leq t \leq b$.

**Solution:**
Given \( K \geq 0 \), let \( \text{Lip}_K \) be the set of functions \( f : \mathbb{R} \to \mathbb{R} \) which satisfy \(|f(x) - f(y)| \leq K|x - y|\) for all \( x, y \in \mathbb{R} \).

(a) Show that the formula

\[
d(f_1, f_2) = \sum_{j=1}^{\infty} 2^{-j} \sup_{z \in [-j, j]} |f_1(z) - f_2(z)|
\]

converges and defines a metric \( d \) on \( \text{Lip}_K \).

(b) Show that \( \text{Lip}_K \) is a complete metric space with this metric.

Solution:
Problem 4A.

Find

\[ \int_{-\infty}^{\infty} \frac{\sin^3(x)}{x^3} \, dx. \]

Solution:
Is there a function $f(z)$ analytic in $\mathbb{C} \setminus \{0\}$ such that $|f(z)| \geq \frac{1}{\sqrt{|z|}}$ for all $z \neq 0$?

Solution:
Problem 6A.

Fix $N \geq 1$. Let $s_1, \ldots, s_N, t_1, \ldots, t_N$ be $2N$ complex numbers of magnitude less than or equal to 1. Let $A$ be the $N \times N$ matrix with entries

$$A_{ij} = \exp (t_is_j).$$

Show that for every $m \geq 1$ there is an $N \times N$ matrix $B$ with rank less than or equal to $m$ such that

$$|A_{ij} - B_{ij}| \leq \frac{2}{m!}$$

for all $i$ and $j$.

Solution:
Problem 7A.

Let $A$ and $B$ be two $n \times n$ matrices with coefficients in $\mathbb{Q}$. For any field extension $K$ of $\mathbb{Q}$, we say that $A$ and $B$ are similar over $K$ if $A = PBP^{-1}$ for some $n \times n$ invertible matrix $P$ with coefficients in $K$. Prove that $A$ and $B$ are similar over $\mathbb{Q}$ if and only if they are similar over $\mathbb{C}$.

Solution:
Let $M_2(\mathbb{Q})$ be the ring of all $2 \times 2$ matrices with coefficients in $\mathbb{Q}$. Describe all field extensions $K$ of $\mathbb{Q}$ such that there is an injective ring homomorphism $K \to M_2(\mathbb{Q})$. (Note: we take the convention that a ring homomorphism maps the multiplicative identity to the multiplicative identity.)

Solution:
Let $p$ be a prime number, $\mathbb{F}_p$ be the finite field of $p$ elements, and $\text{GL}_n(\mathbb{F}_p)$ be the finite group of all invertible $n \times n$ matrices with coefficients in $\mathbb{F}_p$. Find the order of $\text{GL}_n(\mathbb{F}_p)$.

Solution:
Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part B Fall Semester 2016

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______
Problem 1B.  

Let $C = \int_{-\infty}^{\infty} e^{-x^2} \, dx$ and let $S_n$ be the $(n - 1)$-dimensional “surface area” of the unit sphere in $\mathbb{R}^n$ (so $S_2 = 2\pi$, $S_3 = 4\pi/3$).

(a) Prove that $C^n = S_n \Gamma(n/2)/2$, where $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt$.  (Evaluate the integral of $e^{-(x_1^2 + \cdots + x_n^2)}$ over $\mathbb{R}^n$ in rectangular and polar coordinates.)

(b) Show that $s \Gamma(s) = \Gamma(s + 1)$, $\Gamma(1) = 1$.

(c) Evaluate $C$.  (Hint: $S_2 = 2\pi$.)

(d) Evaluate $S_4$.

Solution:
Let $K$ be a compact subset of $\mathbb{R}^n$ and $f(x) = d(x, K)$ be the Euclidean distance from $x$ to the nearest point of $K$.

(a) Show that $f$ is continuous and $f(x) = 0$ if $x \in K$.

(b) Let $g(x) = \max(1 - f(x), 0)$. Show that $\int g^m$ converges to the $n$-dimensional volume of $K$ as $m \to \infty$.

(The $n$-dimensional volume of $K$ is defined to be $\int 1_K$, if the integral exists, where $1_K(x) = 1$ for $x \in K$, and $1_K(x) = 0$ for $x \notin K$.)

Solution:
Problem 3B.  

(a) Suppose that $I$ is a closed interval and $f$ is a smooth function from $I$ to $I$ such that $|f'|$ is bounded by some number $r < 1$ on $I$. Let $a_0$ be in $I$ and put $a_{n+1} = f(a_n)$. Prove that the sequence $a_n$ tends to the unique root of $f(x) = x$ in $I$.

(b) Show that if $a_0$ is real and $a_{n+1} = \cos(a_n)$ then $a_n$ tends to a root of $\cos(x) = x$.

Solution:
Problem 4B.  

Put \( f(z) = z(e^z - 1) \). Prove there exists an analytic function \( h(z) \) defined near \( z = 0 \) such that \( f(z) = h(z)^2 \). Find the first 3 terms in the power series expansion \( h(z) = \sum a_n z^n \). Does \( h(z) \) extend to an entire function on \( \mathbb{C} \)?

Solution:
Let $f_t(z)$ be a family of entire functions depending analytically on $t \in \Delta$, where $\Delta$ is the open unit disk in $\mathbb{C}$. Suppose that for all $t$, $f_t(z)$ is non-vanishing on the unit circle $S^1$ in $\mathbb{C}$. Prove that for each $k \geq 0$,

$$N_k(t) = \sum_{|z|<1; f_t(z)=0} z^k$$

is an analytic function of $t$ (the zeroes of $f_t(z)$ are taken with multiplicity in the sum).

Solution:
Let $A$ be an $m \times n$ matrix of rank $r$ and $B$ a $p \times q$ matrix of rank $s$. Find the dimension of the vector space of $n \times p$ matrices $X$ such that $AXB = 0$.

Solution:
Problem 7B.

Find an example of a vector space $V$ over the real numbers $\mathbb{R}$ and two linear maps $f, g : V \to V$ such that $f$ is injective but not surjective and $g$ is surjective but not injective and such that $f + g$ is equal to the identity map $1_V$.

Hint: construct $V$ as a subspace of the space of sequences of real numbers, closed under the linear maps

$$f(a_1, a_2, a_3, \ldots) = (a_1 - a_2, a_2 - a_3, \ldots)$$

and

$$g(a_1, a_2, a_3, \ldots) = (a_2, a_3, \ldots).$$

Solution:
Let $G$ be a group and $n$ be a positive integer. Assume that there exists a surjective group homomorphism $\mathbb{Z}^n \to G$ and an injective group homomorphism $\mathbb{Z}^n \to G$. Prove that the group $G$ is isomorphic to $\mathbb{Z}^n$.

Solution:
Find (with proof) the number of groups of order 12 up to isomorphism. You may assume
the Sylow theorems (if a prime power $p^n$ is the largest power of $p$ dividing the order of a
 group, then the group has subgroups of order $p^n$ and the number of them is $1 \mod p$.)

Solution: