Problem 1A.  

Show that 
\[ \int_0^1 x^{-x} \, dx = \sum_{n=1}^{\infty} n^{-n} \]

Solution:  
Write \( x^{-x} = e^{-x \log x} \), Taylor expand the exponential, and integrate term by term.

Problem 2A.  

Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable and satisfies \( f'(x) > f(x) \) for all real \( x \). Show that if \( f(0) = 0 \) then \( f(x) > 0 \) for all \( x > 0 \).

Solution:  
Since \( f'(0) > 0 \) we have \( f(x) = x \cdot f'(0) + o(|x|) \) in a neighborhood of zero, so there is a \( t > 0 \) such that \( f \) is positive on \((0, t)\). Assume for contradiction that \( f(x) \leq 0 \) for some \( x > 0 \) and let \( x_0 \) be the first such \( x \). Then \( f(x) > 0 \) on \((0, x_0)\), which means that \( f'(x) > 0 \) on \((0, x_0)\), so \( f(x_0) > f(0) = 0 \), a contradiction.

Problem 3A.  

Let \( X \) be a metric space.
(a) If \( U \) is a subset of \( X \) show that there is a unique open set \( \neg U \) disjoint from \( U \) and containing all open sets disjoint from \( U \).
(b) Give an example of an open set \( U \) with \( U \neq \neg \neg U \)
(c) Prove that for all open sets \( U \), \( \neg U = \neg \neg \neg U \). (Hint: if \( A \subseteq B \) and \( B \subseteq A \) then \( A = B \).)

Solution:  
(a) Take \( \neg U \) to be the union of all open sets disjoint from \( U \), which is open as the union of any collection of open sets is open.
(b) Take \( X \) to be the real line and \( U \) to be the nonzero reals. Then \( \neg U \) is empty so \( \neg \neg U \) is the real line.
(c) We have \( A \subseteq \neg \neg A \) and applying this to \( A = \neg U \) we get \( \neg U \subseteq \neg \neg U \). On the other hand, if \( A \subseteq B \) then \( \neg B \subseteq \neg A \), and applying this to \( A = U, B = \neg U \) we get \( \neg \neg U \subseteq \neg U \).
Combining these gives \(-\neg U = -U\).

**Problem 4A.**

Let \(a\) be a real number with \(|a| < 1\). Prove that

\[
\sum_{k=1}^{\infty} a^k \cos(k\theta) = \frac{-a^2 + a \cos \theta}{1 + a^2 - 2a \cos \theta}
\]

**Solution:** We use the fact that for any complex number \(z = e^{i\theta} = \cos \theta + i \sin \theta \in \mathbb{C}\)

\[
\frac{1}{1 - az} = \sum_{k=0}^{\infty} a^k z^k = \sum_{k=0}^{\infty} a^k e^{ik\theta} = 1 + \sum_{k=1}^{\infty} a^k (\cos(k\theta) + i \sin(k\theta)).
\]

Therefore

\[
\sum_{k=1}^{\infty} a^k \cos(k\theta) = \Re \left( \frac{1}{1 - az} - 1 \right) = \Re \left( \frac{az}{1 - az} \right) = \Re \left( \frac{az(1 - \overline{a} z)}{|1 - az|^2} \right) = \Re \left( \frac{az - a^2}{(1 - a \cos \theta)^2 + (a \sin \theta)^2} \right) = \frac{a \cos \theta - a^2}{1 + a^2 - 2a \cos \theta}
\]

**Problem 5A.**

Describe a conformal map from the set \(\{|z - 4i| < 4\} \cap \{|z - i| > 1\}\) onto the open unit disk.

**Solution:** Compose

\[
f_1 : z \to 1/z \\
f_2 : z \to 8\pi(z + i/2)/3 \\
f_3 : z \to \exp(z) \\
f_4 : z \to (z - i)/(z + i)
\]
Problem 6A.

Let $A$ be an $n \times n$ matrix with real entries such that $(A - I)^m = 0$ for some $m \geq 1$. Prove that there exists an $n \times n$ matrix $B$ with real entries such that $B^2 = A$.

**Solution:** Write $A = I + N$, so $N^m = 0$. Let $P(x)$ be the $m$-th Taylor polynomial of the function $\sqrt{1 + x}$, so $P(x)^2 \equiv 1 + x \pmod{x^m}$. In other words

$$P(x)^2 = 1 + x + x^m Q(x)$$

for some $Q(x) \in \mathbb{R}[x]$. Then

$$P(N)^2 = I + N + N^m Q(N) = I + N = A,$$

so $B := P(N)$ satisfies $B^2 = A$. S

Problem 7A.

Suppose $A = (a_{ij})$ is a real symmetric $n \times n$ matrix with nonnegative eigenvalues. Show that

$$|a_{ij}| \leq \sqrt{a_{ii}a_{jj}}$$

for all distinct $i, j \leq n$.

**Solution:**

Since $A$ is symmetric with nonnegative eigenvalues, we may diagonalize $A$ as $A = UDU^T$ with positive $D$, so $A = B^TB$ for $B^T = UD^{1/2}$. Thus, $A$ is a Gram matrix, i.e., $a_{ij} = \langle v_i, v_j \rangle$ where $v_i$ are the columns of $B$, so by Cauchy Schwartz $a_{ij} \leq \|v_i\|\|v_j\| \leq \sqrt{a_{ii}a_{jj}}$, as desired.

Problem 8A.

For three non-zero integers $a, b$ and $c$ show that

$$\text{gcd}(a, \text{lcm}(b, c)) = \text{lcm}(\text{gcd}(a, b), \text{gcd}(a, c)).$$

where gcd and lcm stand for the greatest common divisor and the least common multiple of two integers, respectively.
Solution: Given a prime $p$, let $\alpha, \beta,$ and $\gamma$ be the exponents of $p$ in the prime factorization of $a, b,$ and $c$, respectively. Then it will suffice to show that

$$\min\{\alpha, \max\{\beta, \gamma\}\} = \max\{\min\{\alpha, \beta\}, \min\{\alpha, \gamma\}\}.$$ 

Without loss of generality, we may assume that $\beta \leq \gamma$; in that case $\max\{\beta, \gamma\} = \gamma$ and $\min\{\alpha, \beta\} \leq \min\{\alpha, \gamma\}$. Therefore the above equation is true because both sides are equal to $\min\{\alpha, \gamma\}$.

Problem 9A.  

Suppose a prime number $p$ divides the order of a finite group $G$. Prove the existence of an element $g \in G$ of order $p$.

Solution: Consider the set $X = \{(g_1, \ldots, g_p) \in G^p \mid g_1 \cdots g_p = e\}$. It is acted upon by the cyclic group $\mathbb{Z}/p\mathbb{Z}$ with $1 \in \mathbb{Z}/p\mathbb{Z}$ acting as the cyclic shift

$$(g_1, \ldots, g_p) \mapsto (g_p, g_1, \ldots, g_{p-1}).$$

A fixed point of this action is a constant $p$-tuple $(g, \ldots, g)$ such that $g^p = e$. The number of fixed points is not zero, since $(e, \ldots, e)$ is a fixed point, and is congruent modulo $p$ to

$$|X| = |G|^{p-1},$$

i.e., it is divisible by $p$, since $p > 1$. It follows that there is an element $g \neq e$ with $g^p = e$.

Problem 1B.  

A mathematician (stupidly) tries to estimate $\pi^2/6 = \sum_{n=1}^{\infty} 1/n^2$ by taking the sum of the first $N$ terms of the series. What is the smallest value of $N$ such that the error of this approximation is at most $10^{-6}$? Hint: integral test.

Solution: The integral test shows that $1/(N + 1) < \sum_{n=N+1}^{\infty} 1/n^2 < 1/N$, so $N = 10^6$.

Problem 2B.  

Suppose $p(z)$ is a nonconstant real polynomial such that for some real number $a$, $p(a) \neq 0$ and $p'(a) = p''(a) = 0$. Prove that $p$ must have at least one nonreal zero.

**Solution:** Observe that if $q(z)$ is a real-rooted polynomial with distinct roots, then by Rolle’s theorem $q'(z)$ is also real-rooted (since it has degree one less than the degree of $q$) and has the property that between every two roots of $q'$ there is a root of $q$. Since polynomials with distinct roots are dense in the set of real-rooted polynomials, this implies that if $q$ is any real-rooted polynomial and $q'(z)$ has a double root at $z$ then $q(z) = 0$.

For the given polynomial $p'(z)$ has a double root at $a$, but $p(a) \neq 0$, so $p$ cannot be real-rooted.

---

**Problem 3B.**

Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets must be monotone.

**Solution:** We prove the contrapositive. Assume $f$ is not monotone, i.e., there exist $a < b < c$ with $f(a) < f(b)$ and $f(b) > f(c)$ or with $f(a) > f(b)$ and $f(b) < f(c)$. In the first case, let $m$ be the point at which $f(x)$ is maximized in $[a, c]$; such a point exists since $f$ is continuous. Moreover we must have $m \neq a, c$ by the hypothesis. But now the image of $(a, c)$ under $f$ contains $m$, but does not contain a neighborhood of $m$, so $f$ cannot map open sets to open sets.

The second case is completely analogous.

---

**Problem 4B.**

Evaluate

$$\int_{-\infty}^{\infty} \frac{x - \sin x}{x^3} \, dx.$$ 

**Solution:**

Integrate by parts twice to reduce to $(1/6) \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, dx$, which is a standard example in complex analysis.

---

**Problem 5B.**

Score:
Suppose $h(z)$ is entire, $h(0) = 3 + 4i$, and $|h(z)| \leq 5$ whenever $|z| < 1$. What is $h'(0)$?

**Solution:** We have $|h(0)| = \sqrt{9 + 16} = 5$, so $|h(0)| \geq |h(z)|$ for $z \in D = \{|z| < 1\}$. By the maximum modulus principle this is only possible if $h(z)$ is constant on $D$, which implies that $h'(0) = 0$.

---

**Problem 6B.**

Score:

Show that if $A$ is an $n \times n$ complex matrix satisfying

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all $i \in \{1, \ldots, n\}$, then $A$ must be invertible.

**Solution:**

Assume $Ax = 0$ and choose $i$ such that $|x_i| = \max_j |x_j|$. Then

$$|a_{ii}| |x_i| \leq \sum_{j \neq i} |a_{ij}| |x_j| \leq \sum_{j \neq i} |a_{ij}| |x_i|$$

so that

$$\left(|a_{ii}| - \sum_{j \neq i} |a_{ij}|\right) |x_i| \leq 0.$$

Since the first factor is positive by assumption and the second is nonnegative, we must have $x_i = 0$. By choice of $i$ we must have $x = 0$ so $A$ is invertible.

---

**Problem 7B.**

Score:

For a real symmetric positive definite matrix $A$ and a vector $v \in \mathbb{R}^n$, show that

$$\int_{\mathbb{R}^n} \exp(-x^T A x + 2v^T x) \, dx = \frac{\pi^{n/2}}{\sqrt{\det A}} \exp(v^T A^{-1} v)$$

You may assume that $\int_{-\infty}^\infty e^{-x^2} \, dx = \sqrt{\pi}$.

**Solution:**

Complete the square, orthogonally diagonalize $A$, change variables, and integrate.
Problem 8B.  

Show that there are no natural numbers $x, y \geq 1$ such that  

$$x^2 + y^2 = 7xy.$$  

**Solution:** Assume that there was such a solution. Taking remainders modulo 7 gives us  

$$x^2 + y^2 \equiv 0 \mod 7.$$  

The quadratic remainders modulo 7 are 0, 1, 2, 4. The only two quadratic remainders whose sum is $\equiv 0$ are 0 and 0. So  

$$x^2 \equiv y^2 \equiv 0 \mod 7.$$  

It follows that $x, y$ are both divisible by 7, i.e. $x = 7x_1$, $y = 7y_1$, for some natural numbers $x_1, y_1$. It follows that  

$$x_1^2 + y_1^2 = 7x_1y_1.$$  

Repeating this process would produce an infinite sequence of pairs $(x, y), (x_1, y_1), (x_2, y_2), \ldots$ such that $x_i$ and $y_i$ are strictly decreasing sequences of integers. Contradiction.

Problem 9B.  

Find the smallest $n$ for which the permutation group $S_n$ contains a cyclic subgroup of order 111.  

**Solution:** Let the partition $n = n_1 + n_2 + \ldots + n_k$ represent the cycle structure of an element $g \in S_n$, i.e. $g$ is a products of commuting cycles of the lengths $n_1 \leq n_2 \leq \ldots \leq n_k$. The order of the cyclic subgroup generated by $g$ is obviously equal to the least common multiple of $n_1, \ldots, n_k$. We want this least common multiple to be 111 = 3 · 37. One of the possibilities is $(n_1, n_2, \ldots, n_k) = (3, 37)$ in which case $n = 3 + 37 = 40$. We claim that this value of $n$ is the minimal possible. Indeed, if 111 is the least common multiple of $n_1, \ldots, n_k$ then each of the prime factors 3, 37 divides at least one of the numbers $n_i$ and moreover, the sum of such factors dividing $n_i$ does not exceed their product and thus does not exceed $n_i$. This implies $n = n_1 + \ldots + n_k \geq 3 + 37 = 40$.  

Score: