FALL 2007 PRELIMINARY EXAMINATION SOLUTIONS

1A. Let $\mathbb{Z}[i]$ be the set of complex numbers of the form $a + bi$ where a and b range over all integers. List all subrings of $\mathbb{Z}[i]$. (Your list should contain each subring exactly once.)

Solution: For $n \in \mathbb{Z}_{\geq 1}$, let $R_n = \mathbb{Z} + n\mathbb{Z}i$. We claim that $\mathbb{Z}$, R_1, R_2, … is a list of all subrings of $\mathbb{Z}[i]$.

First, each R_i is a subring since it contains 0 and 1 and is closed under negation, addition, and multiplication. And of course $\mathbb{Z}$ is a subring too.

Now we show that any subring R equals either $\mathbb{Z}$ or some R_n. Any subring R is an additive subgroup of $\mathbb{Z}[i]$ containing $\mathbb{Z}$. The additive subgroups of $\mathbb{Z}[i]$ containing $\mathbb{Z}$ are the inverse images of subgroups of the quotient group $\mathbb{Z}[i]/\mathbb{Z}$, which is isomorphic to $\mathbb{Z}$ via the homomorphism sending the class of $a + bi$ to b. The subgroups of $\mathbb{Z}$ are $\{0\}$ and $n\mathbb{Z}$ for $n \in \mathbb{Z}_{\geq 1}$, and their inverse images under $\mathbb{Z}[i] \to \mathbb{Z}[i]/\mathbb{Z} \simeq \mathbb{Z}$ are $\mathbb{Z}$ and R_n, respectively.

2A. Let $f(z)$ and $g(z)$ be entire functions such that $f'(z) = g(z)$, $g'(z) = -f(z)$, and $f(2z) = 2f(z)g(z)$ for all $z \in \mathbb{C}$. Find all possibilities for $f(z)$.

Solution: The first two identities imply $f''(z) = -f(z)$, to which the general solution is $f(z) = ae^{iz} + be^{-iz}$ where $a, b \in \mathbb{C}$. Conversely, if $a, b \in \mathbb{C}$, then the functions $f(z) := ae^{iz} + be^{-iz}$ and $g(z) := f'(z) = aie^{iz} - bie^{-iz}$ satisfy the first two identities.

It remains to check which $a, b \in \mathbb{C}$ lead to the third identity being satisfied. The third identity says

$$ae^{2iz} + be^{-2iz} = 2(ae^{iz} + be^{-iz})(aie^{iz} - bie^{-iz})$$

or equivalently,

$$(a - 2a^2 i)e^{4iz} = -b - 2b^2 i.$$

This holds for all $z \in \mathbb{C}$ if and only if $a - 2a^2 i = 0$ and $-b - 2b^2 i = 0$. These equations are equivalent to $a \in \{0, -i/2\}$ and $b \in \{0, i/2\}$. Thus there are four possibilities for $f(z)$, namely 0, $-ie^{iz}/2$, $ie^{-iz}/2$, and

$$-ie^{iz}/2 + ie^{-iz}/2 = \sin z.$$

3A. Let A be an $n \times n$ Hermitian matrix, and let $x \in \mathbb{C}^n$ be a vector such that $A^2 x = 0$. Prove that $Ax = 0$.

Solution: We have: $A^2 x = 0 \Rightarrow A^H A x = 0$ (since $A^H = A$) $\Rightarrow x^H A^H A x = 0 \Rightarrow \|Ax\|^2 = \langle Ax, Ax \rangle = 0 \Rightarrow \|Ax\| = 0 \Rightarrow Ax = 0$.

4A. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers. Suppose that $0 \leq a_{n+1} \leq a_n + b_n$ for all $n \geq 1$, and that $\sum_{n=1}^{\infty} b_n$ converges. Prove that $\lim_{n\to\infty} a_n$ exists and is finite.

Solution: Fix any $\epsilon > 0$. Since $\sum b_n$ converges, there exists $N_\epsilon < \infty$ such that for all $n \geq N_\epsilon$ and all $k \geq 0$, we have $|b_n + b_{n+1} + \cdots + b_{n+k}| < \epsilon$. Hence for all $n \geq N_\epsilon$ and $k \geq 0$,

$$\begin{aligned} a_{n+k+1} &\leq a_n + b_n + \cdots + b_{n+k} \\ &< a_n + \epsilon. \end{aligned}$$

Therefore $\sup_{m>n} a_m \leq a_n + \epsilon$. Hence $\limsup_{n\to\infty} a_n < \infty$.

All the a_n except possibly a_1 are nonnegative, so $\liminf a_n$ is finite. Take $n_1 < n_2 < \ldots$ such that $a_{n_k} \to \liminf a_n$. Then

$$\begin{aligned} \limsup a_n &= \lim_{k\to\infty} \sup_{m>n_k} a_m \\ &\leq \lim_{k\to\infty} (a_{n_k} + \epsilon) \\ &= \epsilon + \liminf a_n. \end{aligned}$$

Sending ϵ to zero shows that $\limsup a_n \leq \liminf a_n$. But $\limsup a_n \geq \liminf a_n$ trivially, so $\limsup a_n = \liminf a_n$. This means that $\lim a_n$ exists and is finite.

5A. Suppose that G is a finite group such that for each subgroup H of G there exists a homomorphism $\phi\colon G \to H$ such that $\phi(h) = h$ for all $h \in H$. Show that G is a product of groups of prime order.

Solution: We proceed by induction on $|G|$. The base case $|G| = 1$ is trivial. Suppose that $|G| > 1$ and that the statement is true for all smaller groups. Choose a subgroup H of G of prime order p. By assumption, there is a homomorphism $\phi\colon G \to H$ such that $\phi(h) = h$ for all $h \in H$. Let $K = \ker\phi$. By the inductive hypothesis, K is a product of groups of prime order. Let $\sigma\colon G \to K$ be a homomorphism such that $\sigma(h) = h$ for all $h \in K$. Let $\alpha\colon G \to K \times H$ be the homomorphism defined by

$$\alpha(g) := (\sigma(g), \phi(g)).$$

Since σ restricted to $\ker\phi$ equals the identity on K, the kernel of α is trivial. Also $|G| = |K||H|$, so α is an isomorphism. The result follows because H has order p.

6A. Let $f(z) = z^4 + \dfrac{z^3}{4} - \dfrac{1}{4}$. How many zeros does f have in $\{z \in \mathbb{C} : \dfrac{1}{2} < |z| < 1\}$?

Solution: We claim that f has 4 zeros in the given annulus. We use Rouché's Theorem at least once. Let $g_1(z) = z^4$. Then g_1 has four zeros (counted with multiplicity) in $\{z \in \mathbb{C} : |z| < 1\}$ and

$$\begin{aligned} |f(z) - g_1(z)| &= \left|\frac{z^3}{4} - \frac{1}{4}\right| \\ &\leq \frac{1}{2} < |g_1(z)| \end{aligned}$$

on $|z| = 1$. Hence f also has four zeros in $\{z \in |z| < 1\}$. There are two ways to proceed from here:

(1) For $|z| \leq \dfrac{1}{2}$, $|f(z)| \geq \dfrac{1}{4} - \dfrac{1}{16} - \dfrac{1}{32} > 0$. Hence f has no zeros in $|z| \leq \dfrac{1}{2}$.

(2) Let $g_2(z) = -3/4$. Then $|f(z) - g_2(z)| \le \frac{1}{16} + \frac{1}{32} + \frac{1}{2} < \frac{3}{4} \equiv |g_2(z)|$ for $|z| = \frac{1}{2}$. Hence f and g_2 have no zeros inside $|z| \le 1/2$.

7A. Let $P \in \mathbb{R}^{n\times n}$ be a matrix satisfying $P^3 = P$. Let r be the rank of P and assume $r > 0$. Show that there exist matrices $U, V \in \mathbb{R}^{n\times r}$ satisfying $V^T U = I_r$ such that

$$P = USV^T,$$

where I_r is the $r \times r$ identity matrix, and S is an $r \times r$ diagonal matrix with ± 1's on the diagonal.

Solution: Since P satisfies the polynomial equation $x^3 - x = 0$ with distinct real roots $0, 1, -1$, the Jordan normal form theorem implies that there exist matrices $T, J \in \mathbb{R}^{n\times n}$ such that $P = TJT^{-1}$ where T is nonsingular and J is diagonal with r nonzero entries. Moreover, we may assume that these r nonzero entries (all ± 1) are in the upper left part of the diagonal of J.

Thus $J = \operatorname{diag}(S, \mathbf{0})$, where S is a $r \times r$ diagonal matrix with ± 1's on the diagonal. Let $U \in \mathbb{R}^{n\times r}$ be the first r columns of T, and let $V \in \mathbb{R}^{n\times r}$ be the transpose of the first r rows of T^{-1}. It follows that $V^T U = I_r$ and $P = USV^T$.

8A. Suppose that $(b_n)_{n\ge 1}$ is a sequence of positive real numbers tending to infinity such that $b_n/n \to 0$. Must there exist a sequence $(a_n)_{n\ge 1}$ such that $(a_1 + \cdots + a_n)/n \to 0$ and $\limsup_{n\to\infty}(a_n/b_n) = \infty$?

Solution: Yes. Replacing b_n with $b_n^* = \max\limits_{1\le k\le n} b_k$, we may suppose that (b_n) is non-decreasing: this does not upset the hypothesis $b_n/n \to 0$. Then there exist $1 \le n_1 < n_2 < \dots$ such that both $\frac{n_{k+1}}{n_k} \to \infty$ and $\frac{b_{n_{k+1}}}{b_{n_k}} \to \infty$ as $k \to \infty$. Let $a_{n_k} = \sqrt{n_k b_{n_k}}$ and let $a_j = 0$ if $j \notin \{n_1, n_2, \dots\}$. For $n_k \le j < n_{k+1}$, we have

$$\left|\frac{a_1 + \cdots + a_j}{j}\right| \le \sum_{i=1}^{k} \frac{|a_{n_i}|}{n_k} \le \frac{(1 + o(1))\sqrt{n_k b_{n_k}}}{n_k},$$

which tends to 0 as $k \to \infty$, while

$$\varlimsup_{n\to\infty} \frac{a_n}{b_n} = \varlimsup_{k\to\infty} \frac{a_{n_k}}{b_{n_k}} = \varlimsup_{k\to\infty} \sqrt{\frac{n_k}{b_{n_k}}} = \infty.$$

9A. Let G be a non-abelian group of order 16 having a subgroup H isomorphic to $C_2 \times C_2 \times C_2$ (where C_2 denotes a cyclic group of order 2). Prove that the number of elements of G of exact order 2 is either 7 or 11.

Solution: Since $(G : H) = 2$, the subgroup H is normal in G. We may regard H as a 3-dimensional vector space over $\mathbb{F}_2$. There are $2^3 - 1 = 7$ elements of order 2 in H.

Case 1: $G - H$ contains no element of order 2. Then the number of order 2 elements of G is also 7.

Case 2: Suppose that $G - H$ contains an element d of order 2. Then G is the semidirect product of $\langle d\rangle$ by H, and is determined up to isomorphism by the conjugation action of d on

H; this action must be nontrivial, since otherwise G would be Abelian. The action is given by an element D of $M_3(\mathbb{F}_2)$ of order 2. In particular the eigenvalues are all 1. A Jordan block of size 3 does not have order 2, so D must consist of Jordan blocks of size 2 and 1. Thus for a suitable choice of basis of H, we have $D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. An element of $G - H$ of order 2 is of the form dh where $(dh)^2 = e$, or equivalently $(dhd^{-1})h = e$; the corresponding values of h are those in the kernel of $D - I$, so there are 4 of them. Thus G has $7 + 4 = 11$ elements of order 2.

1B. Let $f(z)$ be a polynomial with complex coefficients, and let a be a complex number. Prove that $\{a, f(a), f(f(a)), \ldots\}$ is not dense in $\mathbb{C}$.

Solution: Let $S = \{a, f(a), f(f(a)), \ldots\}$. If S is bounded, then S is not dense in $\mathbb{C}$. So assume that S is unbounded.

Case 0: f is constant. Then $\#S \leq 2$, so S is not dense in $\mathbb{C}$.

Case 1: $\deg f = 1$. Write $f(z) = sz + t$ for some $s, t \in \mathbb{C}$ with $s \neq 0$. If $s = 1$, then S is contained in a line, and hence is not dense. So suppose that $s \neq 1$. Then $f(z) = z$ has a solution $z = c$, and replacing $f(z)$ by $f(z+c) - c$ (and replacing S by $-c + S$) lets us reduce to the case where $t = 0$. Now $S = \{a, sa, s^2a, \ldots\}$. Since S is unbounded, $|s| > 1$. But then S contains only finitely many points in each disk, so S is not dense in $\mathbb{C}$.

Case 2: $\deg f \geq 2$. Then $f(z)/z \to \infty$ as $z \to \infty$, so there exists $M > 0$ such that $|z| > M$ implies $|f(z)| > |z|$. Since S is unbounded, there exists n such that $|f^n(a)| > M$. By induction, we obtain $|f^N(a)| > M$ for all $N \geq n$. Thus S contains only finitely many points in the disk $|z| \leq M$, so S is not dense in $\mathbb{C}$.

2B. Let A be an $n \times n$ complex matrix. Suppose that m is a positive integer such that A^m is diagonalizable. Prove that A^{m+1} is diagonalizable.

Solution: We may assume that A is in Jordan canonical form, and we may reduce to the case where A is a single Jordan block, so $A = \lambda I + N$, where $\lambda \in \mathbb{C}$ and N is nilpotent.

Case 1: $\lambda = 0$. Then N^m is nilpotent and diagonalizable, so $N^m = 0$. Hence $N^{m+1} = N \cdot 0 = 0$.

Case 2: $\lambda \neq 0$. Then A^m is diagonalizable with all eigenvalues equal to λ^m, so $A^m = \lambda^m I$. In particular A satisfies the equation $x^m - \lambda^m = 0$ with distinct roots, so A is diagonalizable. Thus A^{m+1} is diagonalizable.

3B. Let $\{u_1, u_2, \cdots, u_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, and let $\mathcal{A}$ be a closed set in $\mathbb{R}^k$. Let S be the set of linear combinations $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k$ obtained as $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ ranges over all points of $\mathcal{A}$. Show that S is a closed subset of $\mathbb{R}^n$.

Solution: Extend $u_1, \ldots, u_k$ to a basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$, and let U be the $n \times n$ matrix whose columns are the u_i. Since U is invertible, it induces a homeomorphism of $\mathbb{R}^n$.

Let $\mathbf{0}$ be the origin in $\mathbb{R}^{n-k}$. Then $A \times \{\mathbf{0}\}$ is closed in $\mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, and S is the image of $A \times \{\mathbf{0}\}$ under the homeomorphism $U \colon \mathbb{R}^n \to \mathbb{R}^n$, so S is closed.

4B. Let K and L be fields, and let $K \times L$ be the product ring, with addition and multiplication defined componentwise. Find all prime ideals of $K \times L$.

Solution: The first projection $K \times L \to K$ is surjective and its kernel is an ideal I such that $(K \times L)/I$ is a field (isomorphic to K), so I is a maximal ideal. Similarly the kernel of the second projection is a maximal ideal J.

Now let P be any prime ideal of $K \times L$. Since $(1,0)(0,1) = 0$, either $(1,0)$ or $(0,1)$ is in P. If $(1,0) \in P$, then $(a,0) = (a,0)(1,0) \in P$ for all a, so $J \subseteq P$; but J is maximal, so then $P = J$. Similarly if $(0,1) \in P$, then $P = I$.

Thus I and J are the only prime ideals of $K \times L$.

5B. Let $f(z)$ be an entire function and let $a_1, \dots, a_n$ be all zeros of f in $\mathbb{C}$. Suppose that there exist real numbers $R > 0$ and $a > 1$ such that $|f(z)| \geq |z|^a$ for all $|z| \geq R$. Prove that

$$\sum_{j=1}^{n} \mathrm{Res}_{z=a_j} \frac{1}{f(z)} = 0.$$

Solution: Let $g(z) = 1/f(z)$. Let $R_0 > R$ be large enough that all $a_1, \dots, a_n$ are inside the circle $|z| = R_0$. Let $r \geq R_0$. We have

$$\int_{|z|=r} g(z)dz = 2\pi i \sum_{j=1}^{n} \mathrm{Res}(g, a_j).$$

This is true for *all* $r \geq R_0$. Also

$$\begin{aligned} \left| \int_{|z|=r} g(z)dz \right| &= \left| ir \int_0^{2\pi} \frac{dt}{f(re^{it})} e^{it} \right| \leq \\ r \int_0^{2\pi} \frac{dt}{|f(re^{it})|} &\leq 2\pi r \frac{1}{r^a} = \frac{2\pi}{r^{a-1}}. \end{aligned}$$

Thus

$$\left| \sum_{j=1}^{n} \mathrm{Res}(g, a_j) \right| \leq \frac{2\pi}{r^{a-1}} \text{ for all } r \geq R_0.$$

Hence $\sum_{j=1}^{n} \mathrm{Res}(g, a_j) = 0$, since $\frac{1}{r^{a-1}} \to 0$ where $r \to \infty$, since $a > 1$.

6B. Given a positive integer n, what are the possible values of the triple $(\mathrm{rk}(A), \mathrm{rk}(B), \mathrm{rk}(C))$ as A, B, C range over real $n \times n$ matrices satisfying $A + B + C = 0$?

Solution: We claim that the answer is the set of triples (a, b, c) of integers in $[0, n]$ satisfying $c \leq a + b$, $a \leq b + c$, and $b \leq c + a$.

The image of C is contained in the sum of the images of A and B, so $\mathrm{rk}(C) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$. Similarly, $\mathrm{rk}(A) \leq \mathrm{rk}(B) + \mathrm{rk}(C)$ and $\mathrm{rk}(B) \leq \mathrm{rk}(C) + \mathrm{rk}(A)$.

Conversely, suppose that a, b, c satisfy the inequalities. Without loss of generality, $c \geq a, b$. Let A be the diagonal matrix whose diagonal entries are a ones followed by $n - a$ zeros. Let B be the diagonal matrix whose diagonal entries are $c - b$ zeros followed by b ones followed by $n - c$ zeros. Let $C := -(A + B)$, so $A + B + C = 0$. Then $\mathrm{rk}(A) = a$, $\mathrm{rk}(B) = b$, and $\mathrm{rk}(C) = \mathrm{rk}(A + B) = c$, since C is a diagonal matrix with exactly c nonzero entries.

7B. Let f be continuous on $[0, \infty)$ and suppose that $\lim_{x \to \infty} f(x)$ exists and is finite. Must f be uniformly continuous? Give a proof or a counterexample.

Solution: The function f is uniformly continuous. Given $\epsilon > 0$, we must find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. There exists $x_0 \geq 0$ such that for all $x \geq x_0$, we have $|f(x) - L| < \epsilon/3$. By compactness of $[0, x_0]$ there exists $\delta > 0$ such that for all x, y in $[0, x_0]$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/3$; choose such a δ.

Suppose that $0 \leq x \leq y < x + \delta$; we must prove that $|f(x) - f(y)| < \epsilon$. If $y \leq x_0$ we are done. If $x \leq x_0 < y$ then by the triangle inequality,

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - L| + |L - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

Finally, if $x_0 \leq x$ then $|f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \epsilon$. Thus f is uniformly continuous.

8B. Show that for every positive integer n, there exists an irreducible polynomial over $\mathbb{Q}$ of degree n such that all its roots are real.

Solution: Let N be a large positive integer. Let $f(x) = \prod_{k=1}^{n}(x - 2^N k)$ and $g(x) = 2 + f(x)$. Then $g(x)$ is irreducible by Eisenstein's criterion. Also

$$\lim_{x \to \infty} g(x) = \infty, \text{ and } \lim_{x \to -\infty} g(x) = (-1)^n \infty.$$

Let $1 \leq j \leq n - 1$ be an integer.

$$f(2^N(j + 1/2)) = 2^{Nn} \prod_{k=1}^{n} (j + 1/2 - k)$$

$$\prod_{k=1}^{n} (j + 1/2 - k) = \frac{-1}{4} \prod_{k=1}^{j-1} (j + 1/2 - k) \prod_{k=j+2}^{n} (j + 1/2 - k).$$

Thus

$$|f(2^N(j + 1/2))| > 2^{Nn-2} \text{ and } \operatorname{sgn}(f(2^N(j + 1/2))) = (-1)^{n-j}.$$

It follows that for large N (actually $N \geq 2$ will work), $g(x)$ has a zero in $(-\infty, 2^N(1 + 1/2))$, in $(2^N(j + 1/2), 2^N(j + 3/2))$ for $1 \leq j \leq n - 2$ and in $(2^N(n - 1/2, \infty))$. Thus g has n real roots.

9B. Let f be holomorphic on a neighborhood of the closed disk $\overline{B_1(0)} = \{z : |z| \leq 1\}$. Suppose that $\max_{|z|=1} |f(z)| \leq 1$. Prove that there exists a complex number z such that $|z| \leq 1$ and $f(z) = z$.

Solution: Let $\alpha_n > 1$ and $\alpha_n \to 1$. Let $g_n(z) = f(z) - \alpha_n z$ and $h_n(z) = \alpha_n z$. Then $|g_n(z) + h_n(z)| = |f(z)| \leq 1 < \alpha_n = |h_n(z)|$ for all $|z| = 1$. By Rouché's Theorem there is z_n with $|z_n| < 1$ such that $g_n(z_n) = 0$ or $f(z_n) = \alpha_n z_n$, $n = 1, 2, \ldots$. Let z be a limit point of $\{z_n\}$, i.e., $z = \lim_{k \to \infty} z_{n_k}$ for some subsequence $\{z_{n_k}\}$ of $\{z_n\}$. Then $|z| \leq 1$ and $f(z) = \lim(\alpha_{n_k} z_{n_k}) = (\lim \alpha_{n_k})(\lim z_{n_k}) = 1 \cdot z = z$.