FALL 2007 PRELIMINARY EXAMINATION SOLUTIONS

1A. Let \( \mathbb{Z}[i] \) be the set of complex numbers of the form \( a + bi \) where \( a \) and \( b \) range over all integers. List all subrings of \( \mathbb{Z}[i] \). (Your list should contain each subring exactly once.)

Solution: For \( n \in \mathbb{Z}_{\geq 1} \), let \( R_n = \mathbb{Z} + n\mathbb{Z}i \). We claim that \( \mathbb{Z}, R_1, R_2, \ldots \) is a list of all subrings of \( \mathbb{Z}[i] \).

First, each \( R_i \) is a subring since it contains 0 and 1 and is closed under negation, addition, and multiplication. And of course \( \mathbb{Z} \) is a subring too.

Now we show that any subring \( R \) equals either \( \mathbb{Z} \) or some \( R_n \). Any subring \( R \) is an additive subgroup of \( \mathbb{Z}[i] \) containing \( \mathbb{Z} \). The additive subgroups of \( \mathbb{Z}[i] \) containing \( \mathbb{Z} \) are the inverse images of subgroups of the quotient group \( \mathbb{Z}[i]/\mathbb{Z} \), which is isomorphic to \( \mathbb{Z} \) via the homomorphism sending the class of \( a + bi \) to \( b \). The subgroups of \( \mathbb{Z} \) are \( \{0\} \) and \( n\mathbb{Z} \) for \( n \in \mathbb{Z}_{\geq 1} \), and their inverse images under \( \mathbb{Z}[i] \to \mathbb{Z}[i]/\mathbb{Z} \simeq \mathbb{Z} \) are \( \mathbb{Z} \) and \( R_n \), respectively.

2A. Let \( f(z) \) and \( g(z) \) be entire functions such that \( f'(z) = g(z) \), \( g'(z) = -f(z) \), and \( f(2z) = 2f(z)g(z) \) for all \( z \in \mathbb{C} \). Find all possibilities for \( f(z) \).

Solution: The first two identities imply \( f''(z) = -f(z) \), to which the general solution is \( f(z) = ae^{iz} + be^{-iz} \) where \( a, b \in \mathbb{C} \). Conversely, if \( a, b \in \mathbb{C} \), then the functions \( f(z) := ae^{iz} + be^{-iz} \) and \( g(z) := f'(z) = aie^{iz} - bie^{-iz} \) satisfy the first two identities.

It remains to check which \( a, b \in \mathbb{C} \) lead to the third identity being satisfied. The third identity says

\[
ae^{2iz} + be^{-2iz} = 2( ae^{iz} + be^{-iz})(aie^{iz} - bie^{-iz})
\]

or equivalently,

\[
(a - 2a^2i)e^{4iz} = -b - 2b^2i.
\]

This holds for all \( z \in \mathbb{C} \) if and only if \( a - 2a^2i = 0 \) and \(-b - 2b^2i = 0 \). These equations are equivalent to \( a \in \{0, -i/2\} \) and \( b \in \{0, i/2\} \). Thus there are four possibilities for \( f(z) \), namely \( 0, -ie^{iz}/2, ie^{-iz}/2 \), and

\[
-ie^{iz}/2 + ie^{-iz}/2 = \sin z.
\]

3A. Let \( A \) be an \( n \times n \) Hermitian matrix, and let \( x \in \mathbb{C}^n \) be a vector such that \( A^2 x = 0 \). Prove that \( Ax = 0 \).

Solution: We have: \( A^2 x = 0 \Rightarrow A^H A x = 0 \) (since \( A^H = A \)) \( \Rightarrow \ x^H A^H A x = 0 \) \( \Rightarrow \ ||Ax||^2 = \langle Ax, Ax \rangle = 0 \Rightarrow ||Ax|| = 0 \Rightarrow Ax = 0 \).

4A. Let \( (a_n)_{n \geq 1} \) and \( (b_n)_{n \geq 1} \) be sequences of real numbers. Suppose that \( 0 \leq a_{n+1} \leq a_n + b_n \) for all \( n \geq 1 \), and that \( \sum_{n=1}^{\infty} b_n \) converges. Prove that \( \lim_{n \to \infty} a_n \) exists and is finite.
Solution: Fix any $\epsilon > 0$. Since $\sum b_n$ converges, there exists $N_\epsilon < \infty$ such that for all $n \geq N_\epsilon$ and all $k \geq 0$, we have $|b_n + b_{n+1} + \cdots + b_{n+k}| < \epsilon$. Hence for all $n \geq N_\epsilon$ and $k \geq 0$, 
\[ a_{n+k+1} \leq a_n + b_n + \cdots + b_{n+k} < a_n + \epsilon. \]
Therefore $\sup_{m>n} a_m \leq a_n + \epsilon$. Hence $\limsup_{n \to \infty} a_n < \infty$.

All the $a_n$ except possibly $a_1$ are nonnegative, so $\liminf a_n$ is finite. Take $n_1 < n_2 < \ldots$ such that $a_{n_k} \to \liminf a_n$. Then 
\[
\limsup a_n = \limsup_{k \to \infty} \sup_{m>n_k} a_m
\leq \lim_{k \to \infty} (a_{n_k} + \epsilon)
= \epsilon + \liminf a_n.
\]
Sending $\epsilon$ to zero shows that $\limsup a_n \leq \liminf a_n$. But $\limsup a_n \geq \liminf a_n$ trivially, so $\limsup a_n = \liminf a_n$. This means that $\lim a_n$ exists and is finite.

5A. Suppose that $G$ is a finite group such that for each subgroup $H$ of $G$ there exists a homomorphism $\phi: G \to H$ such that $\phi(h) = h$ for all $h \in H$. Show that $G$ is a product of groups of prime order.

Solution: We proceed by induction on $|G|$. The base case $|G| = 1$ is trivial. Suppose that $|G| > 1$ and that the statement is true for all smaller groups. Choose a subgroup $H$ of $G$ of prime order $p$. By assumption, there is a homomorphism $\phi: G \to H$ such that $\phi(h) = h$ for all $h \in H$. Let $K = \ker \phi$. By the inductive hypothesis, $K$ is a product of groups of prime order. Let $\sigma: G \to K$ be a homomorphism such that $\sigma(h) = h$ for all $h \in K$. Let $\alpha: G \to K \times H$ be the homomorphism defined by 
\[
\alpha(g) := (\sigma(g), \phi(g)).
\]
Since $\sigma$ restricted to $\ker \phi$ equals the identity on $K$, the kernel of $\alpha$ is trivial. Also $|G| = |K||H|$, so $\alpha$ is an isomorphism. The result follows because $H$ has order $p$.

6A. Let $f(z) = z^4 + \frac{z^3}{4} - \frac{1}{4}$. How many zeros does $f$ have in $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$?

Solution: We claim that $f$ has 4 zeros in the given annulus. We use Rouché’s Theorem at least once. Let $g_1(z) = z^4$. Then $g_1$ has four zeros (counted with multiplicity) in $\{z \in \mathbb{C} : |z| < 1\}$ and 
\[
|f(z) - g_1(z)| = \left| \frac{z^3}{4} - \frac{1}{4} \right|
\leq \frac{1}{2} < |g_1(z)|
\]
on $|z| = 1$. Hence $f$ also has four zeros in $\{z \in |z| < 1\}$. There are two ways to proceed from here:

(1) For $|z| \leq \frac{1}{2}$, $|f(z)| \geq \frac{1}{4} - \frac{1}{16} - \frac{1}{32} > 0$. Hence $f$ has no zeros in $|z| \leq \frac{1}{2}$. 

(2) For $|z| > \frac{1}{2}$, $|f(z)|$ is dominated by $|g_1(z)|$. Thus $f$ has no zeros in $|z| > \frac{1}{2}$. 

Therefore, $f$ has exactly 4 zeros in the annulus $\{z \in \mathbb{C} : \frac{1}{2} < |z| < 1\}$. 


(2) Let \( g_2(z) = -3/4 \). Then \( |f(z) - g_2(z)| \leq \frac{1}{16} + \frac{1}{32} + \frac{1}{2} < \frac{3}{4} \equiv |g_2(z)| \) for \( |z| = \frac{1}{2} \).

Hence \( f \) and \( g_2 \) have no zeros inside \(|z| \leq 1/2\).

7A. Let \( P \in \mathbb{R}^{n \times n} \) be a matrix satisfying \( P^3 = P \). Let \( r \) be the rank of \( P \) and assume \( r > 0 \). Show that there exist matrices \( U, V \in \mathbb{R}^{n \times r} \) satisfying \( V^T U = I_r \) such that

\[
P = U S V^T,
\]

where \( I_r \) is the \( r \times r \) identity matrix, and \( S \) is an \( r \times r \) diagonal matrix with \( \pm 1 \)'s on the diagonal.

Solution: Since \( P \) satisfies the polynomial equation \( x^3 - x = 0 \) with distinct real roots \( 0, 1, -1 \), the Jordan normal form theorem implies that there exist matrices \( T, J \in \mathbb{R}^{n \times n} \) such that \( P = T J T^{-1} \) where \( T \) is nonsingular and \( J \) is diagonal with \( r \) nonzero entries. Moreover, we may assume that these \( r \) nonzero entries (all \( \pm 1 \)) are in the upper left part of the diagonal of \( J \).

Thus \( J = \text{diag}(S, 0) \), where \( S \) is a \( r \times r \) diagonal matrix with \( \pm 1 \)'s on the diagonal. Let \( U \in \mathbb{R}^{n \times r} \) be the first \( r \) columns of \( T \), and let \( V \in \mathbb{R}^{n \times r} \) be the transpose of the first \( r \) rows of \( T^{-1} \). It follows that \( V^T U = I_r \) and \( P = U S V^T \).

8A. Suppose that \( (b_n)_{n \geq 1} \) is a sequence of positive real numbers tending to infinity such that \( b_n/n \to 0 \). Must there exist a sequence \( (a_n)_{n \geq 1} \) such that \( (a_1 + \cdots + a_n)/n \to 0 \) and \( \limsup_{n \to \infty} (a_n/b_n) = \infty \)?

Solution: Yes. Replacing \( b_n \) with \( b_n^* = \max_{1 \leq k \leq n} b_k \), we may suppose that \( (b_n) \) is non-decreasing: this does not upset the hypothesis \( b_n/n \to 0 \). Then there exist \( 1 \leq n_1 < n_2 < \cdots \) such that both \( n_{k+1}/n_k \to \infty \) and \( b_{n_{k+1}}/b_{n_k} \to \infty \) as \( k \to \infty \). Let \( a_{n_k} = \sqrt{n_k b_{n_k}} \) and let \( a_j = 0 \) if \( j \notin \{n_1, n_2, \ldots\} \). For \( n_k \leq j < n_{k+1} \), we have

\[
\left| \frac{a_1 + \cdots + a_j}{j} \right| \leq \sum_{i=1}^{k} \frac{|a_{n_i}|}{n_k} \leq \frac{(1 + o(1)) \sqrt{n_k b_{n_k}}}{n_k},
\]

which tends to 0 as \( k \to \infty \), while

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{k \to \infty} \frac{a_{n_k}}{b_{n_k}} = \lim_{k \to \infty} \sqrt{\frac{n_k}{b_{n_k}}} = \infty.
\]

9A. Let \( G \) be a non-abelian group of order 16 having a subgroup \( H \) isomorphic to \( C_2 \times C_2 \times C_2 \) (where \( C_2 \) denotes a cyclic group of order 2). Prove that the number of elements of \( G \) of exact order 2 is either 7 or 11.

Solution: Since \( (G : H) = 2 \), the subgroup \( H \) is normal in \( G \). We may regard \( H \) as a 3-dimensional vector space over \( \mathbb{F}_2 \). There are \( 2^3 - 1 = 7 \) elements of order 2 in \( H \).

Case 1: \( G - H \) contains no element of order 2. Then the number of order 2 elements of \( G \) is also 7.

Case 2: Suppose that \( G - H \) contains an element \( d \) of order 2. Then \( G \) is the semidirect product of \( \langle d \rangle \) by \( H \), and is determined up to isomorphism by the conjugation action of \( d \) on
$H$; this action must be nontrivial, since otherwise $G$ would be Abelian. The action is given by an element $D$ of $M_3(\mathbb{F}_2)$ of order 2. In particular the eigenvalues are all 1. A Jordan block of size 3 does not have order 2, so $D$ must consist of Jordan blocks of size 2 and 1.

Thus for a suitable choice of basis of $H$, we have $D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. An element of $G - H$ of order 2 is of the form $dh$ where $(dh)^2 = e$, or equivalently $(dh^{-1})h = e$; the corresponding values of $h$ are those in the kernel of $D - I$, so there are 4 of them. Thus $G$ has $7 + 4 = 11$ elements of order 2.

1B. Let $f(z)$ be a polynomial with complex coefficients, and let $a$ be a complex number. Prove that \{a, f(a), f(f(a)), \ldots\} is not dense in $\mathbb{C}$.

Solution: Let $S = \{a, f(a), f(f(a)), \ldots\}$. If $S$ is bounded, then $S$ is not dense in $\mathbb{C}$. So assume that $S$ is unbounded.

Case 0: $f$ is constant. Then $\# S \leq 2$, so $S$ is not dense in $\mathbb{C}$.

Case 1: $\deg f = 1$. Write $f(z) = sz + t$ for some $s, t \in \mathbb{C}$ with $s \neq 0$. If $s = 1$, then $S$ is contained in a line, and hence is not dense. So suppose that $s \neq 1$. Then $f(z) = z$ has a solution $z = c$, and replacing $f(z)$ by $f(z + c) - c$ (and replacing $S$ by $-c + S$) lets us reduce to the case where $t = 0$. Now $S = \{a, sa, s^2a, \ldots\}$. Since $S$ is unbounded, $|s| > 1$. But then $S$ contains only finitely many points in each disk, so $S$ is not dense in $\mathbb{C}$.

Case 2: $\deg f \geq 2$. Then $f(z)/z \to \infty$ as $z \to \infty$, so there exists $M > 0$ such that $|z| > M$ implies $|f(z)| > |z|$. Since $S$ is unbounded, there exists $n$ such that $|f^n(a)| > M$. By induction, we obtain $|f^N(a)| > M$ for all $N \geq n$. Thus $S$ contains only finitely many points in the disk $|z| \leq M$, so $S$ is not dense in $\mathbb{C}$.

2B. Let $A$ be an $n \times n$ complex matrix. Suppose that $m$ is a positive integer such that $A^n$ is diagonalizable. Prove that $A^{m+1}$ is diagonalizable.

Solution: We may assume that $A$ is in Jordan canonical form, and we may reduce to the case where $A$ is a single Jordan block, so $A = \lambda I + N$, where $\lambda \in \mathbb{C}$ and $N$ is nilpotent.

Case 1: $\lambda = 0$. Then $N^m$ is nilpotent and diagonalizable, so $N^m = 0$. Hence $N^{m+1} = N \cdot 0 = 0$.

Case 2: $\lambda \neq 0$. Then $A^n$ is diagonalizable with all eigenvalues equal to $\lambda^n$, so $A^m = \lambda^m I$. In particular $A$ satisfies the equation $x^m - \lambda^m = 0$ with distinct roots, so $A$ is diagonalizable. Thus $A^{m+1}$ is diagonalizable.

3B. Let $\{u_1, u_2, \ldots, u_k\}$ be a set of linearly independent vectors in $\mathbb{R}^n$, and let $\mathcal{A}$ be a closed set in $\mathbb{R}^k$. Let $S$ be the set of linear combinations $\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k$ obtained as $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ ranges over all points of $\mathcal{A}$. Show that $S$ is a closed subset of $\mathbb{R}^n$.

Solution: Extend $u_1, \ldots, u_k$ to a basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$, and let $U$ be the $n \times n$ matrix whose columns are the $u_i$. Since $U$ is invertible, it induces a homeomorphism of $\mathbb{R}^n$.

Let $0$ be the origin in $\mathbb{R}^{n-k}$. Then $A \times \{0\}$ is closed in $\mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, and $S$ is the image of $A \times \{0\}$ under the homeomorphism $U : \mathbb{R}^n \to \mathbb{R}^n$, so $S$ is closed.
4B. Let $K$ and $L$ be fields, and let $K \times L$ be the product ring, with addition and multiplication defined componentwise. Find all prime ideals of $K \times L$.

Solution: The first projection $K \times L \to K$ is surjective and its kernel is an ideal $I$ such that $(K \times L)/I$ is a field (isomorphic to $K$), so $I$ is a maximal ideal. Similarly the kernel of the second projection is a maximal ideal $J$.

Now let $P$ be any prime ideal of $K \times L$. Since $(1, 0)(0, 1) = 0$, either $(1, 0)$ or $(0, 1)$ is in $P$. If $(1, 0) \in P$, then $(a, 0) = (a, 0)(1, 0) \in P$ for all $a$, so $J \subseteq P$; but $J$ is maximal, so then $P = J$. Similarly if $(0, 1) \in P$, then $P = I$.

Thus $I$ and $J$ are the only prime ideals of $K \times L$.

5B. Let $f(z)$ be an entire function and let $a_1, \ldots, a_n$ be all zeros of $f$ in $\mathbb{C}$. Suppose that there exist real numbers $R > 0$ and $a > 1$ such that $|f(z)| \geq |z|^a$ for all $|z| \geq R$. Prove that

$$\sum_{j=1}^{n} \text{Res}_{z=a_j} \frac{1}{f(z)} = 0.$$ 

Solution: Let $g(z) = 1/f(z)$. Let $R_0 > R$ be large enough that all $a_1, \ldots, a_n$ are inside the circle $|z| = R_0$. Let $r \geq R_0$. We have

$$\int_{|z|=r} g(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}(g, a_j).$$

This is true for all $r \geq R_0$. Also

$$\left| \int_{|z|=r} g(z)dz \right| = \left| ir \int_{0}^{2\pi} \frac{dt}{f(re^{it})} e^{it} \right| \leq 2\pi r \frac{1}{r^a} = \frac{2\pi}{r^{a-1}}.$$ 

Thus

$$\left| \sum_{j=1}^{n} \text{Res}(g, a_j) \right| \leq \frac{2\pi}{r^{a-1}}$$ 

for all $r \geq R_0$.

Hence $\sum_{j=1}^{n} \text{Res}(g, a_j) = 0$, since $\frac{1}{r^{a-1}} \to 0$ where $r \to \infty$, since $a > 1$.

6B. Given a positive integer $n$, what are the possible values of the triple $(\text{rk}(A), \text{rk}(B), \text{rk}(C))$ as $A, B, C$ range over real $n \times n$ matrices satisfying $A + B + C = 0$?

Solution: We claim that the answer is the set of triples $(a, b, c)$ of integers in $[0, n]$ satisfying $c \leq a + b$, $a \leq b + c$, and $b \leq c + a$.

The image of $C$ is contained in the sum of the images of $A$ and $B$, so $\text{rk}(C) \leq \text{rk}(A) + \text{rk}(B)$. Similarly, $\text{rk}(A) \leq \text{rk}(B) + \text{rk}(C)$ and $\text{rk}(B) \leq \text{rk}(C) + \text{rk}(A)$.

Conversely, suppose that $a, b, c$ satisfy the inequalities. Without loss of generality, $c \geq a, b$.

Let $A$ be the diagonal matrix whose diagonal entries are $a$ ones followed by $n-a$ zeros. Let $B$ be the diagonal matrix whose diagonal entries are $b$ ones followed by $n-b$ zeros. Let $C := -(A + B)$, so $A + B + C = 0$. Then $\text{rk}(A) = a$, $\text{rk}(B) = b$, and $\text{rk}(C) = \text{rk}(A + B) = c$, since $C$ is a diagonal matrix with exactly $c$ nonzero entries.
7B. Let $f$ be continuous on $[0, \infty)$ and suppose that $\lim_{x \to \infty} f(x)$ exists and is finite. Must $f$ be uniformly continuous? Give a proof or a counterexample.

Solution: The function $f$ is uniformly continuous. Given $\epsilon > 0$, we must find $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. There exists $x_0 \geq 0$ such that for all $x \geq x_0$, we have $|f(x) - L| < \epsilon/3$. By compactness of $[0, x_0]$ there exists $\delta > 0$ such that for all $x, y$ in $[0, x_0]$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon/3$; choose such a $\delta$.

Suppose that $0 \leq x \leq y < x + \delta$; we must prove that $|f(x) - f(y)| < \epsilon$. If $y \leq x_0$ we are done. If $x < x_0 < y$ then by the triangle inequality,

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - L| + |L - f(y)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$ 

Finally, if $x_0 \leq x$ then $|f(x) - f(y)| \leq |f(x) - L| + |L - f(y)| < \epsilon$. Thus $f$ is uniformly continuous.

8B. Show that for every positive integer $n$, there exists an irreducible polynomial over $\mathbb{Q}$ of degree $n$ such that all its roots are real.

Solution: Let $N$ be a large positive integer. Let $f(x) = \prod_{k=1}^{n}(x - 2^{Nk})$ and $g(x) = 2 + f(x)$. Then $g(x)$ is irreducible by Eisenstein’s criterion. Also

$$\lim_{x \to \infty} g(x) = \infty,$$

and

$$\lim_{x \to -\infty} g(x) = (-1)^n \infty.$$

Let $1 \leq j \leq n - 1$ be an integer.

$$f(2^N(j + 1/2)) = 2^{Nn} \prod_{k=1}^{n}(j + 1/2 - k)$$

$$\prod_{k=1}^{n}(j + 1/2 - k) = -1 \frac{j-1}{4} \prod_{k=1}^{n}(j + 1/2 - k) \prod_{k=j+2}^{n}(j + 1/2 - k).$$

Thus

$$|f(2^N(j + 1/2))| > 2^{Nn-2} \text{ and } \text{sgn}(f(2^N(j + 1/2))) = (-1)^{n-j}.$$ 

It follows that for large $N$ (actually $N \geq 2$ will work), $g(x)$ has a zero in $(-\infty, 2^N(1 + 1/2))$, in $2^N(j + 1/2, 2^N(j + 3/2))$ for $1 \leq j \leq n - 2$ and in $2^N(n - 1/2, \infty))$. Thus $g$ has $n$ real roots.

9B. Let $f$ be holomorphic on a neighborhood of the closed disk $\overline{B_1(0)} = \{z : |z| \leq 1\}$. Suppose that $\max_{|z| = 1} |f(z)| \leq 1$. Prove that there exists a complex number $z$ such that $|z| \leq 1$ and $f(z) = z$.

Solution: Let $\alpha_n > 1$ and $\alpha_n \to 1$. Let $g_n(z) = f(z) - \alpha_n z$ and $h_n(z) = \alpha_n z$. Then $|g_n(z) + h_n(z)| = |f(z)| \leq 1 < \alpha_n = |h_n(z)|$ for all $|z| = 1$. By Rouché’s Theorem there is $z_n$ with $|z_n| < 1$ such that $g_n(z_n) = 0$ or $f(z_n) = \alpha_n z_n$, $n = 1, 2, \ldots$. Let $z$ be a limit point of $\{z_n\}$, i.e., $z = \lim_{k \to \infty} z_n$ for some subsequence $\{z_{n_k}\}$ of $\{z_n\}$. Then $|z| \leq 1$ and $f(z) = \lim (\alpha_n z_{n_k}) = (\lim \alpha_{n_k})(\lim z_{n_k}) = 1 \cdot z = z$. 

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