FALL 2006 PRELIMINARY EXAMINATION

1A. Compute

$$\lim_{x \to 0} \frac{d^4}{dx^4} \frac{x}{\sin x}.$$

2A. Let

$$A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

Compute

$$e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

3A. Let U be a connected open subset of $\mathbb{C}$ containing -2 and 0. Suppose that $f\colon U \to \mathbb{C}$ is a holomorphic function whose Taylor expansion at 0 is $\sum_{n \geq 0} \binom{2n}{n} z^n$. Prove that $f(-2) \in \{1/3, -1/3\}$. (Note: The original version of this problem had an error: $\{3, -3\}$ instead of $\{1/3, -1/3\}$.)

4A. Let R be a finite commutative ring without zero-divisors and containing at least one element other than 0. (As usual, rings are associative with 1.) Prove that R is a field.

5A. Let $C^0[0,1]$ be the vector space over $\mathbb{R}$ consisting of continuous functions from $[0,1]$ to $\mathbb{R}$. Show that the linear operator $T\colon C^0[0,1] \to C^0[0,1]$ defined by

$$(Tf)(x) := \int_0^x f(y)\, dy$$

has no nonzero eigenvectors.

6A. Let p be prime. Prove that the polynomial $f(x) = x^p - x + 1$ is irreducible over the field $\mathbb{F}_p$ of p elements.

7A. Prove that for every $a \in \mathbb{C}$ and integer $n \geq 2$, the equation $1 + z + az^n = 0$ has at least one root in the disk $|z| \leq 2$.

8A. Let $\mathbf{Z}$ denote the ring of integers and consider the linear map $\mathbf{Z}^3 \to \mathbf{Z}^3$ defined by the 3×3-matrix

$$A = \begin{pmatrix} 6 & 9 & 12 \\ 6 & 9 & 12 \\ 12 & 18 & 24 \end{pmatrix}$$

Compute the structure of the three abelian groups $\operatorname{kernel}(A)$, $\operatorname{image}(A)$, and $\operatorname{cokernel}(A) = \mathbf{Z}^3/\operatorname{image}(A)$. In particular, in each case determine whether the group is free abelian. If yes, give a basis.

9A. Let k be a field such that the additive group of k is finitely generated. Prove that k is finite.

1B. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume that $|f(z^2)| \leq 2|f(z)|$ for all $z \in \mathbb{C}$. Show that f is constant.

2B. Let $C^0[0,1]$ be the vector space over $\mathbb{R}$ consisting of continuous functions from $[0,1]$ to $\mathbb{R}$. Show that the functions $1, x, x^2, \ldots$ are linearly independent in $C^0[0,1]$.

3B. Let $f\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ be a continuous function. For $x \in \mathbb{R}$, define

$$g(x) := \max\{f(x,y) : y \in [0,1]\}.$$

Show that g is continuous.

4B. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial. Suppose there is a field extension F of $\mathbb{Q}$ containing a root a of $f(x)$ such that F does not contain any cube root of a. Show that $f(x^3)$ is irreducible over $\mathbb{Q}$.

5B. Let f and g be entire functions such that

$$\int_{|z|=1} \frac{f(z)}{(\sin z)^m}\, dz = \int_{|z|=1} \frac{g(z)}{(\sin z)^m}\, dz$$

for all positive integers m. Prove that $f = g$.

6B. Let G be a nonabelian group of order 21. Find the largest positive integer n with the property that whenever G acts on a set S of size n, some element of S is fixed by every element of G.

7B. Let X and Y be metric spaces, and let $f_1, f_2, \ldots$ be continuous functions from X to Y. Suppose that the sequence $\{f_n\}$ converges uniformly to a function f. Show that f is continuous.

8B. Let A be an $n \times n$ Hermitian matrix and B an $n \times n$ positive definite (complex) matrix. Prove that there is an invertible complex $n \times n$ matrix S such that $S^H AS$ is diagonal and $S^H BS = I$. (Here S^H denotes the conjugate transpose of the matrix S.)

9B. Let $z_0, z_1, \ldots$ be a sequence of complex numbers such that $z_{n+1} = 1 + 1/z_n$ for all $n \geq 0$. Prove that the sequence is convergent.