FALL 2005 PRELIMINARY EXAMINATION

1A. Let M be a compact metric space and let $(U_i)_{i\in I}$ be an open cover of M. Show that there exists $\varepsilon > 0$ such that, for all $x, y \in M$, if $d(x,y) < \varepsilon$ then there is some j with both x and y in U_j.

2A. Prove that, if $f(z) = P(z)/Q(z)$ is a rational function with complex coefficients whose numerator has lower degree than the denominator, then $f(z)$ is a sum of terms of the form $a/(z-b)^k$, with $a, b \in \mathbb{C}$.

3A. Define $U \subseteq \mathbb{C}$ to be the open right half plane with the interval $(0,1] \subseteq \mathbb{R}$ deleted. Find an explicit conformal equivalence of U with the open unit disk D.

4A. Let m and n be positive integers. Prove that the ideal generated by $x^m - 1$ and $x^n - 1$ in $\mathbb{Z}[x]$ is principal.

5A. Is there a differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(0) = 1$ and $f'(x) \geq f(x)^2$ for all $x \in \mathbb{R}$?

6A. Let A be an $n \times n$ matrix with real entries such that $(A - I)^m = 0$ for some $m \geq 1$. Prove that there exists an $n \times n$ matrix B with real entries such that $B^2 = A$.

7A. Let $f(z) = z^5 + 5z^3 + z^2 + z + 1$. How many zeros (counting multiplicity) does f have in the annulus $1 \leq |z| \leq 2$?

8A. Find the smallest n for which the permutation group S_n contains a cyclic subgroup of order 111.

9A. A doubly infinite sequence $(a_j)_{j\in\mathbb{Z}}$ of real numbers is said to be **rapidly decreasing** if, for each positive integer n, the sequence $j^n a_j$ is bounded. Let (a_j) and (b_j) be rapidly decreasing sequences, and define the convolution of these sequences by $c_j = \sum_{k\in\mathbb{Z}} a_k b_{j-k}$ for $j \in \mathbb{Z}$. Prove that the series defining each c_j is convergent, and that (c_j) is a rapidly decreasing sequence.

1B. How many pairs of integers (a, b) are there satisfying $a \geq b \geq 0$ and $a^2 + b^2 = 5 \cdot 17 \cdot 37$?

2B. Let f be a continuous real-valued function defined on $[0, \infty)$, each that $f(x) \geq 0$, f is non-increasing, and $\lim_{x\to\infty} f(x) = 0$. Show that

$$\lim_{R\to\infty} \int_0^R f(x) \sin x \; dx$$

exists. (In other words, the improper integral

$$\int_0^\infty f(x) \sin x \; dx$$

converges.)

3B. For which pairs of monic polynomials $(p(x), m(x))$ over the complex numbers does there exist a matrix in $M_{n,n}(\mathbb{C})$ whose characteristic polynomial is $p(x)$ and whose minimal polynomial is $m(x)$?

4B. Determine which numbers $a \in \mathbb{C}$ have the following property: There exists an analytic function f defined in the open unit disk such that, for all integers $n \geq 2$,

$$f(1/n) = 1/(n+a).$$

5B. Given a prime number p, let $\mathbb{F}_p$ be the field of p elements, and let R be the ring $\mathbb{F}_p[x]/(x^3)$. For which primes p is the unit group R^* cyclic?

6B. Let $K \subset \mathbb{R}^n$ be closed, convex, and nonempty. (*Convex* means that if $x, y \in K$ and $\lambda \in [0,1]$ then $\lambda x + (1-\lambda)y \in K$.) Show that for every $x \in \mathbb{R}^n$, there exists $y \in K$ that uniquely minimizes the Euclidean distance to x, i.e. $\|x - y\| < \|x - z\|$ for all $z \in K \setminus \{y\}$.

7B. Let V be a finite-dimensional complex vector space equipped with a positive-definite Hermitian inner product. Let $T\colon V \to V$ be a Hermitian (i.e., self-adjoint) linear operator. Prove that

(a) $1 + iT$ is nonsingular (where $i = \sqrt{-1}$); and
(b) $(1 - iT)(1 + iT)^{-1}$ is a unitary operator.

8B. For $R > 0$ let Γ_R be the semicircle $\{|z| = R,\ \text{Im} z \geq 0\}$ (radius R, center 0, in the upper half-plane). Prove that

$$\lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{z} dz = 0.$$

9B. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function. Prove that there exists a countable subfield K of $\mathbb{R}$ such that $f(K) \subseteq K$.