FALL 2004 PRELIMINARY EXAMINATION

1A. Show that there is a unique piecewise continuous function $y(x)$ on $\mathbb{R}$ satisfying the two conditions

$$\begin{aligned} y(x) &= \int_0^\infty e^{-2s} y(x-s)\, ds && \text{for } x > 0, \text{ and} \\ y(x) &= e^x, && \text{for } x \le 0, \end{aligned}$$

and find an explicit formula for $y(x)$ for $x > 0$.

2A. For $c \in \mathbb{Q}$, define $R_c := \mathbb{Q}[x]/(x^3 - cx)$. Let $a, b \in \mathbb{Q}$. Show that the rings R_a and R_b are isomorphic if and only if there exists a nonzero $r \in \mathbb{Q}$ such that $b = r^2 a$.

3A. Let f and g be functions that are holomorphic on all of $\mathbb{C}$, except that g has an essential singularity at the complex number c. Prove that either f is constant, or the composition $f \circ g$ has an essential singularity at c. (Hint: you may assume the Casorati-Weierstrass Theorem, which states that if a function f has an essential singularity at c, then for any punctured neighborhood N of c on which f is holomorphic, the image $f(N)$ is dense in $\mathbb{C}$.)

4A. Let A be an $n \times n$ matrix with complex entries. Prove that A is diagonalizable if and only if the following is true: Whenever f is a polynomial with complex coefficients such that $f(A)$ is nilpotent, we have $f(A) = 0$. (A matrix A is *nilpotent* if $A^m = 0$ for some $m \ge 1$.)

5A. Let $(a_m)_{m\ge 1}$ be a sequence of real numbers satisfying $a_{n+m} \le a_n + a_m$. Prove that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}$$

as an element of $[-\infty, \infty)$.

6A. Let n be a square-free positive integer (*i.e.*, $n = 1$ or n is prime or n is a product of distinct primes). Assume that for every product of primes $pq_1 \cdots q_r$ dividing n, with $r > 0$, we have $q_1 \cdots q_r \not\equiv 1 \pmod p$. Prove that every group G of order n is abelian.

7A. Let D be the open unit disk in $\mathbb{C}$, and $f \colon D \to D$ a holomorphic function. Suppose that $f(-\frac{1}{2}) = 0$ and $f(0) = \frac{1}{2}$. Prove that there is only one possible value for $f(\frac{1}{2})$, and find it.

8A. Let $\langle\ ,\ \rangle$ be a positive-definite Hermitian inner product on a finite-dimensional complex vector space V. Suppose $T \colon V \to V$ is a $\mathbb{C}$-linear map such that $\langle Tv, v\rangle = 0$ for all $v \in V$. Prove that $T = 0$.

9A. Let $f \colon [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 f(x) e^{inx^3} dx = 0.$$

1B. Let S_n be the group of permutations of $\{1, \ldots, n\}$, and let A_n be the alternating subgroup. Suppose $m \leq n$.

(a) Identify S_m with the subgroup of S_n consisting of elements that fix $m+1$, ..., n. Prove that $A_n \cap S_m = A_m$.

(b) Is it true in general that if $f \colon S_m \to S_n$ is an injective homomorphism, then $A_n \cap f(S_m) = f(A_m)$? Give a proof or a counterexample.

2B. Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be a continuous function of compact support (i.e., f vanishes outside some bounded set).

(a) Show that

$$u(x) := \int \frac{f(y)}{|x-y|}\, dy$$

converges, where the integral is over all $y \in \mathbb{R}^3$.

(b) Show that $\lim_{|x|\to\infty} u(x)|x|$ exists.

3B. For which positive integers n does there exist an $n \times n$ matrix A with rational entries such that $A^3 + A + I = 0$?

4B. Evaluate $I(w) := \displaystyle\int_0^\infty \frac{e^{iwt}}{\sqrt{t}}\, dt$ for every nonzero real number w. You may use the formula $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$.

5B. What is the cardinality of the smallest field F of characteristic 7 such that the equation $x^{18} + x^{17} + \cdots + x + 1 = 0$ has a solution $x \in F$?

6B. Suppose that $f(z)$ is holomorphic on all of $\mathbb{C}$ except for a pole at $z = 0$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{1}{n} e^{2\pi i k/n}\right)$$

exists.

7B. Let $n \geq 1$, and let $M_n(\mathbb{R})$ be the ring of $n \times n$ matrices over the field of real numbers. What is the dimension of the subspace V of $M_n(\mathbb{R})$ spanned by the matrices of the form $AB - BA$ where $A, B \in M_n(\mathbb{R})$?

8B. A C^2 function $y(x)$ for $0 \leq x \leq 1$, a positive continuous function $a(x)$ for $0 \leq x \leq 1$, and a real number λ satisfy

$$\begin{aligned} y''(x) + \lambda a(x) y(x) &= 0, \\ y(0) &= 0, \\ y'(1) &= 0. \end{aligned}$$

Suppose that $y(x)$ is not identically zero. Prove that $\lambda > 0$.

9B. Prove that every group of order 30 has a cyclic subgroup of order 15.