1A. Show that the differential equation
\[ f''(z) = zf(z), \quad f(0) = 1, \quad f'(0) = 1 \]
has an unique entire solution in the complex plane.

Solution. Let
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]
be the Taylor series of \( f \) at 0. Then the equation gives
\[ a_0 = 1, \quad a_1 = 1, \quad a_2 = 0 \]
\[ k(k - 1)a_k = a_{k-3}. \]
Hence for \( k \geq 1 \) we obtain
\[ a_{3k} = \prod_{j=1}^{k} \frac{1}{3j(3j - 1)} \]
\[ a_{3k+1} = \prod_{j=1}^{k} \frac{1}{3j(3j + 1)} \]
\[ a_{3k+2} = 0. \]
We need to show that the convergence radius of the series for \( f \) is infinite. Indeed we have
\[ \lim_{k \to \infty} \frac{a_{3k+3}}{a_{3k}} = 0 \]
which shows that the series
\[ \sum_{k=0}^{\infty} a_{3k} z^{3k} \]
has an infinite radius of convergence. Similarly we argue for the “3k + 1” series.

2A. List eight groups of order 36 and prove that they are not isomorphic.

Solution. Let \( C_n \) be a cyclic group of order \( n \), let \( D_{2n} \) be a dihedral group of order \( 2n \), let \( S_n \) be the symmetric group on \( n \) letters, and let \( A_n \) be its alternating subgroup. Consider the following eight groups of order 36:

\[ C_2^2 \times C_3^2, \quad C_2^2 \times C_9, \quad C_4 \times C_3^2, \quad C_4 \times C_9, \quad C_6 \times S_3, \quad S_3 \times S_3, \quad C_2 \times D_{2,9}, \quad C_3 \times A_4. \]

The first four are abelian and pairwise nonisomorphic because each pair has either distinct 2-Sylow subgroups or distinct 3-Sylow subgroups. They are not isomorphic to the last four because the latter are nonabelian.

Of the last four, only \( C_2 \times D_{2,9} \) has a cyclic 3-Sylow subgroup, only \( C_3 \times A_4 \) has a normal 2-Sylow subgroup, and only \( S_3 \times S_3 \) has a trivial center. Thus the last four also are pairwise nonisomorphic.

(Remark: in fact, there are 14 groups of order 36.)
3A. Let $A$ be a $2 \times 2$ matrix with complex entries. Prove that the series $I + A + A^2 + \ldots$ converges if and only if every eigenvalue of $A$ has absolute value less than 1.

Solution. Conjugating $A$ changes neither the convergence nor the eigenvalues, so we may assume that $A$ is in Jordan canonical form, i.e., $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ or $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

In the first case, $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ and $\sum A^n$ converges if and only if the eigenvalues $a$ and $b$ have absolute value less than 1, because the entries of the sum are geometric series.

In the second case, write $A = aI + N$, so $N^2 = 0$, and $A^n = a^nI + na^{n-1}N$. If $I + A + A^2 + \ldots$ converges, then the diagonal entries $a^n$ of the terms $A^n$ must converge to 0, so $|a| < 1$. Conversely if $|a| < 1$, then $\sum a^n$ and $\sum na^{n-1}$ converge by the Ratio Test, so $\sum A^n$ converges.

4A. Give an example, with proof, of a nonconstant irreducible polynomial $f(x)$ over $\mathbb{Q}$ with the property that $f(x)$ does not factor into linear factors over the field $K = \mathbb{Q}[x]/(f(x))$.

Solution. The simplest example is $f(x) = x^3 - 2$. Let $\sqrt[3]{2}$ denote the real cube root of 2. Then $\mathbb{Q}(\sqrt[3]{2})$ is an algebraic extension of $\mathbb{Q}$ generated by a root of $x^3 - 2$, hence isomorphic to $K = \mathbb{Q}[x]/(x^3 - 2)$. Since $\mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$, and $x^3 - 2$ has only one real root, $x^3 - 2$ does not factor completely over $K$. The same proof works with $f(x) = x^3 - a$ for any rational $a$ that is not a cube of a rational number. Other examples are also possible, of course.

5A. Let $C$ denote the space of continuous functions on $[0, 1]$. Define

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx.$$ 

(a) Show that $d$ is a metric on $C$.

(b) Show that $(C, d)$ is not a complete metric space.

Solution. The function $a \mapsto a/(1 + a) = 1 - 1/(1 + a)$ is increasing on $[0, \infty)$. Hence, for $a = |f - g|$, $b = |g - h|$, $c = |f - h|$, we have $c \leq a + b$ and

$$\frac{c}{1 + c} \leq \frac{a + b}{1 + a + b} = \frac{a}{1 + a + b} + \frac{b}{1 + a + b} \leq \frac{a}{1 + a} + \frac{b}{1 + b}.$$ 

This implies the triangle inequality.

Define

$$f_n(x) = \begin{cases} n^2x, & 0 \leq x \leq 1/n \\ 1/x, & 1/n \leq x \leq 1. \end{cases}$$

The $f_n$ form a Cauchy sequence, since

$$d(f_m, f_n) = \int_0^{\max\{1/m, 1/n\}} \frac{|f_m(x) - f_n(x)|}{1 + |f_m(x) - f_n(x)|} \, dx$$ 

$$\leq \int_0^{\max\{1/m, 1/n\}} \frac{1}{1 + x} \, dx$$ 

$$= \max\{1/m, 1/n\}.$$
Suppose that $(C, d)$ is a complete metric space. Then the $f_n$ would converge to some $f \in C$. If $f(a) \neq 1/a$ for some $a \in (0, 1]$, then by continuity there exists $\epsilon > 0$ such that $|1/x - f(x)| \geq \epsilon$ for $x \in (a - \epsilon, a]$. Then

$$d(f_n, f) \geq \int_{a-\epsilon}^{a} \frac{\epsilon}{1+\epsilon} \, dx$$

for sufficiently large $n$. But the right hand side is a positive constant independent of $n$, so then $f_n$ could not converge to $f$. Thus $f(a) = 1/a$ for all $a \in (0, 1]$. This contradicts the fact that $f$ is continuous on $[0, 1]$.

6A. Let $A(m, n)$ be the $m \times n$ matrix with entries

$$a_{ij} = j^i \quad (0 \leq i \leq m - 1, \ 0 \leq j \leq n - 1),$$

where $0^0 = 1$ by definition. Regarding the entries of $A(m, n)$ as representing congruence classes (mod $p$), determine the rank of $A(m, n)$ over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for all $m, n \geq 1$ and all primes $p$.

Solution. The upper-left $k \times k$ square minor $A(k, k)$ of $A(m, n)$ is the Vandermonde matrix, with determinant $\prod_{0 \leq i < j < k} (j - i)$. If $k \leq p$, this determinant is non-zero (mod $p$), which shows that $\text{rk} A(m, n) \geq \min(m, n, p)$. Conversely, $A(m, n)$ has at most $p$ distinct columns (mod $p$), so $\text{rk} A(m, n) \leq p$. Since $\text{rk} A(n, n) \leq \min(m, n)$, we have $\text{rk} A(m, n) = \min(m, n, p)$.

7A. Let $D = \{z \in \mathbb{C} : |z| \leq 1\} - \{1, -1\}$. Find an explicit continuous function $f : D \to \mathbb{R}$ satisfying all the following conditions:

- $f$ is harmonic on the interior of $D$ (the open unit disk),
- $f(z) = 1$ when $|z| = 1$ and $\text{Im}(z) > 0$, and
- $f(z) = -1$ when $|z| = 1$ and $\text{Im}(z) < 0$.

Solution. The linear fractional transformation $w = (1+z)/(1-z)$ maps $|z| < 1$ to the half-plane $\text{Re}(w) > 0$, with the upper and lower boundary semicircles mapping to the half-lines $i\mathbb{R}_{>0}$ and $i\mathbb{R}_{<0}$, respectively. A branch of $\log w$ defined on $\mathbb{C} - \mathbb{R}_{\leq 0}$ has

$$\text{Im}(\log w) = \begin{cases} \pi/2, & w \in i\mathbb{R}_{>0} \\ -\pi/2, & w \in i\mathbb{R}_{<0}, \end{cases}$$

so $f(z) = \frac{2}{\pi} \text{Im}((1+z)/(1-z))$ is a solution.

8A. Let $p$ be a prime, and let $G$ be the group $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. How many automorphisms does $G$ have?

Solution. An automorphism of $G$ is determined by where it sends the generators $(1,0)$ and $(0,1)$. We claim that for $(a,b), (c,d) \in G$, there exists an automorphism mapping $(1,0)$ to $(a,b)$ and $(0,1)$ to $(c,d)$ if and only if

$$a \notin p\mathbb{Z}/p^2\mathbb{Z}, \quad c \in p\mathbb{Z}/p^2\mathbb{Z}, \quad \text{and} \quad d \neq 0 \in \mathbb{Z}/p\mathbb{Z}.$$
If $\alpha$ is an automorphism mapping $(1, 0)$ to $(a, b)$ and $(0, 1)$ to $(c, d)$, then $(a, b)$ must not be killed by $p$, so $a \not\in p\mathbb{Z}/p^2\mathbb{Z}$ and $(c, d)$ must be killed by $p$, so $c \in p\mathbb{Z}/p^2\mathbb{Z}$. Moreover $(c, d)$ should not be a multiple of $p(a, b) = (pa, 0)$, so $d \neq 0$.

Conversely, given $(a, b)$ and $(c, d)$ satisfying the conditions, there exists a homomorphism $\alpha : G \to G$ mapping $(1, 0)$ to $(a, b)$ and $(0, 1)$ to $(c, d)$, since $(a, b)$ is killed by $p^2$ and $(c, d)$ is killed by $p$. The condition on $\alpha$ implies that $(a, b)$ has order $p^2$. If $(c, d)$ were a multiple of $(a, b)$, then since $c \in p\mathbb{Z}/p^2\mathbb{Z}$, the element $(c, d)$ would be a multiple of $p(a, b) = (pa, 0)$, which is impossible, since $d \neq 0 \in \mathbb{Z}/p\mathbb{Z}$. Thus $\#\alpha(G) > p^2$. So by Lagrange’s theorem $\#\alpha(G) = p^3$. Thus $\alpha$ is surjective, but $G$ is finite, so $\alpha$ is also injective, so $\alpha$ is an automorphism.

It remains to count $(a, b, c, d)$ satisfying the conditions. There are $p^2 - p$ possibilities for $a$, $p$ possibilities for $b$, $p$ possibilities for $c$, and $p - 1$ possibilities for $d$, and these may be chosen independently, so in total there are $(p^2 - p)p^2(p - 1) = p^5 - 2p^4 + p^3$ automorphisms of $G$.

9A. Let $f : [0, 1] \to [0, 1]$ be an increasing (not strictly increasing) function such that

$$f\left(\sum_{j=1}^{\infty} a_j 3^{-j}\right) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$$

whenever the $a_j$ are 0 or 2. Prove that there is a constant $C_0$ such that

$$|f(x) - f(y)| \leq C_0 |x - y|^{(\log 2)/(\log 3)}$$

for all $x, y \in [0, 1]$.

Solution. Let $x = 0.a_1a_2\ldots$ in base 3. If $a_j = 1$ for some $j$, choose the smallest such $j$, and define

$$x_- = 0.a_1a_2\ldots a_{j-1}022222\ldots$$
$$x_+ = 0.a_1a_2\ldots a_{j-1}200000\ldots$$

(These are the nearest numbers in $C$ on either side of $x$, where $C$ is the Cantor set consisting of numbers in $[0, 1]$ representable by base-3 expansions with only 0’s and 2’s.) Then $f(x_-) = f(x_+)$, so $f$ is constant on $[x_-, x_+]$.

Thus it suffices to prove the inequality with $x = \sum a_j 3^{-j} \geq y = \sum b_j 3^{-j}$ with $a_j, b_j \in \{0, 2\}$. Let $\hat{j}$ be the smallest $j$ with $a_j \neq b_j$. Then $|x - y| \geq 3^{-\hat{j}}$. On the other hand,

$$|f(x) - f(y)| = \left|\sum_{j \geq \hat{j}} \frac{a_j - b_j}{2} 2^{-j}\right| \leq \sum_{j \geq \hat{j}} 2^{-j} = 2 \cdot 2^{-\hat{j}}.$$

Combining, we obtain

$$|f(x) - f(y)| \leq 2 \cdot 2^{-\hat{j}} \leq 2(3^{-\hat{j}})^{(\log 2)/(\log 3)} \leq 2|x - y|^{(\log 2)/(\log 3)}.$$
1B. Evaluate \( \int_{-\infty}^{\infty} \frac{x^2}{x^n + 1} \, dx \), where \( n \geq 4 \) is an even integer.

Solution. Let \( f(x) \) be the integrand. The answer is \( 2I \), where \( I := \int_{0}^{\infty} f(x) \, dx \). For \( R > 1 \), let \( \gamma_R \) be the straight line path from 0 to \( R \), followed by the arc \( Re^{it} \) for \( t \in [0, 2\pi/n] \), followed by the straight line path from \( Re^{2\pi i/n} \) back to 0.

Let \( \zeta = e^{\pi i/n} \). The poles of \( f(z) \) are at \( \zeta^{m+1} \) for \( m \in \mathbb{Z} \), so the only pole inside \( \gamma_R \) is \( \zeta \). The numerator is nonzero at \( \zeta \), while the denominator has nonzero derivative at \( \zeta \), so \( \zeta \) is a simple pole with residue

\[
\frac{\zeta^2}{n\zeta^{n-1}} = \frac{1}{n\zeta^{3-n}}.
\]

By the residue theorem,

\[
\int_{\gamma_R} f(z) \, dz = \frac{2\pi i}{n} \zeta^{3-n} = -\frac{2\pi i}{n} \zeta^3.
\]

On the other hand, the first straight part of the integral tends to \( I \) as \( R \to \infty \), the curved part of the integral tends to 0 as \( R \to \infty \) since the integrand is \( O(1/R^{n-2}) \leq O(1/R^2) \) while the length of the arc is \( O(R) \), and the last straight part of the integral tends to \( -\zeta^6 I \) as \( R \to \infty \), as the substitution \( z = \zeta^2 t \) shows. Thus

\[
I - \zeta^6 I = -\frac{2\pi i}{n} \zeta^3.
\]

Now

\[
\sin(3\pi/n) = \frac{\zeta^3 - \zeta^{-3}}{2i} = \frac{\zeta^6 - 1}{2i\zeta^3},
\]

so

\[
2I = \frac{4\pi i}{n} \cdot \frac{\zeta^3}{\zeta^6 - 1}
\]

\[
= \frac{4\pi i}{n} \cdot \frac{1}{2i \sin(3\pi/n)}
\]

\[
= \frac{2\pi}{n \sin(3\pi/n)}.
\]

2B. Let \( u_{m,n} \) be an array of numbers for \( 1 \leq m \leq N \) and \( 1 \leq n \leq N \). Suppose that \( u_{m,n} = 0 \) when \( m \) is 1 or \( N \), or when \( n \) is 1 or \( N \). Suppose also that

\[
u_{m,n} = \frac{1}{4} (u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1})
\]

whenever \( 1 < m < N \) and \( 1 < n < N \). Show that all the \( u_{m,n} \) are zero.

Solution. If not, then by changing signs, we may assume that \( M := \max u_{m,n} \) is positive. Let

\( R = \{(m, n) : u_{m,n} = M\} \subseteq \{2, 3, \ldots, N-1\} \times \{2, 3, \ldots, N-1\} \).

Choose \((m, n) \in R \) with \( m \) minimal. Since \( (m-1, n) \notin R \),

\[
\frac{1}{4} (u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1}) < \frac{1}{4} (M + M + M + M) = M = u_{m,n}.
\]

This contradicts the given relation.
3B. Let $A$ and $B$ be $n \times n$ complex unitary matrices. Prove that $|\det(A + B)| \leq 2^n$.

Solution. Let $C = A^{-1}B$, which also is unitary. Then

$$A + B = A(I + C)$$

Since $A$ is unitary, its eigenvalues have absolute value 1. Multiplying them together shows that $|\det A| = 1$. If $\zeta_1, \ldots, \zeta_n$ are the eigenvalues of $C$ with multiplicity, so $|\zeta_i| = 1$, then the eigenvalues of $I + C$ are $1 + \zeta_1, \ldots, 1 + \zeta_n$, so

$$|\det(I + C)| = |1 + \zeta_1| \ldots |1 + \zeta_n| \leq 2 \cdot 2 \ldots 2 = 2^n$$

Thus

$$|\det(A + B)| = |\det(A)||\det(I + C)| \leq 2^n.$$

4B. Let $L$ be a line in $\mathbb{C}$, and let $f$ be an entire function such that $f(\mathbb{C}) \cap L = \emptyset$. Prove that $f$ is constant. (Do not use the theorem of Picard that the image of a nonconstant entire function omits at most one complex number.)

Solution. Replacing $f$ by $f + c$ for some $c \in \mathbb{C}$, we may assume that $0 \in L$. Replacing $f$ by $\alpha f$ for some $\alpha \in \mathbb{C}^*$, we may assume that $L$ is the imaginary axis. Since $f(\mathbb{C})$ is connected, it is contained in either the right half plane or the left half plane. Replace $f$ by $-f$ if necessary, to assume that $f(\mathbb{C})$ is contained in the left half plane. Then $g(z) = e^{f(z)}$ is entire and bounded, hence it is a constant $c$ by Liouville’s theorem. Then $f(\mathbb{C})$ is contained in the set of solutions to $e^z = c$, which is discrete, but $f(\mathbb{C})$ is connected, so $f(\mathbb{C})$ must be a point. Thus $f$ is constant.

5B. Let $n$ be a positive integer. Let $\phi(n)$ be the Euler phi function, so $\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^*$. Prove that if $\gcd(n, \phi(n)) > 1$, then there exists a noncyclic group of order $n$.

Solution. Let $p$ be a prime dividing both $n$ and $\phi(n)$. The formula for $\phi(n)$ shows that either $p^2|n$ or there is a different prime $q|n$ such that $p|(q - 1)$.

If $p^2|n$, then $C_p \times C_p \times C_{n/p^2}$ is a noncyclic group of order $n$ (where $C_m$ denotes a cyclic group of order $m$).

In the other case, let $G$ be the subgroup of $\text{GL}_2(\mathbb{F}_q)$ consisting of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a^p = 1$. Since $\mathbb{F}_q^*$ is cyclic of order $q - 1$, there are $p$ solutions to $a^p = 1$ in $\mathbb{F}_q$. Thus $\#G = pq$. If $a^p = 1$ and $a \neq 1$, then

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

so $G$ is not abelian. Then $G \times C_{n/pq}$ has order $n$ and is not cyclic (since it is not abelian).

6B. Let $f(z)$ be a meromorphic function on the complex plane. Suppose that for every polynomial $p(z) \in \mathbb{C}[z]$ and every closed contour $\Gamma$ avoiding the poles of $f$, we have

$$\int_{\Gamma} p(z)^2 f(z) \, dz = 0.$$

Prove that $f(z)$ is entire.
Solution. Comparing the condition with \( p(z) \) replaced by \( p(z) + 1 \) and subtracting, we find that
\[
\int_\Gamma (2p(z) + 1)f(z)\,dz = 0.
\]
Every polynomial can be written as \( 2p(z) + 1 \), so we have that
\[
\int_\Gamma p(z)f(z)\,dz = 0
\]
for every polynomial \( p(z) \).

Suppose that \( f(z) \) has a pole of order \( n \) at \( a \in \mathbb{C} \). Then \( (z - a)^{n-1}f(z) \) has a nonzero residue at \( a \), so
\[
\int_\Gamma (z - a)^{n-1}f(z)\,dz \neq 0
\]
for a sufficiently small loop \( \Gamma \) around \( a \). Thus \( f(z) \) cannot have any poles. Hence \( f(z) \) is entire.

7B. (a) Let \( G \) be a finite group and let \( X \) be the set of pairs of commuting elements of \( G \):
\[
X = \{(g, h) \in G \times G : gh = hg\}.
\]
Prove that \(|X| = c|G|\) where \( c \) is the number of conjugacy classes in \( G \).

(b) Compute the number of pairs of commuting permutations on five letters.

Solution. (a) Let \( C_g \) denote the conjugacy class of \( g \) and \( Z_g \) the centralizer of \( g \). By the orbit-stabilizer theorem, we have \(|Z_g| \cdot |C_g| = |G|\) for every \( g \). Hence \( \sum_{g \in C} |Z_g| = |G| \) for every conjugacy class \( C \), and \(|X| = \sum_{g \in G} |Z_g| = c|G|\).

(b) Take \( G = S_5 \), with \(|G| = 5! = 120\). The number of conjugacy classes \( c \) is the number of partitions of 5, namely 7. So there are \( 7 \cdot 120 = 840 \) pairs of commuting permutations.

8B. The set of \( 5 \times 5 \) complex matrices \( A \) satisfying \( A^3 = A^2 \) is a union of conjugacy classes. How many conjugacy classes?

Solution. A matrix \( A \) is a solution to \( x^3 = x^2 \) (or equivalently, \( x^2(x - 1) = 0 \)) if and only if all its Jordan blocks are. In particular, each Jordan block must have eigenvalues 0 and 1, and the possible Jordan blocks are
\[
(0), \quad (1), \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

The conjugacy type of a matrix is determined by the multiplicities of the Jordan blocks. Let \( a, b, c \) be the multiplicities of the blocks above, respectively. Then the answer is the number of nonnegative integer solutions to
\[
a + b + 2c = 5.
\]
For fixed \( c \in \{0, 1, 2\} \), there are \( 6 - 2c \) solutions to \( a + b = 5 - 2c \). Thus the answer is
\[
(6 - 2 \cdot 0) + (6 - 2 \cdot 1) + (6 - 2 \cdot 2) = 12.
\]
9B. Let \( \lambda, a \in \mathbb{R} \), with \( a > 0 \). Let \( u(x, y) \) be an infinitely differentiable function defined on an open neighborhood of \( x^2 + y^2 \leq 1 \) such that

\[
\Delta u + \lambda u = 0 \quad \text{in } x^2 + y^2 < 1 \\
u_n = -au \quad \text{on } x^2 + y^2 = 1.
\]

Here \( \Delta \) is the Laplacian \( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), and \( u_n \) denotes the directional derivative of \( u \) in the direction of the outward unit normal (pointing away from the origin). Prove that if \( u \) is not identically zero in \( x^2 + y^2 < 1 \), then \( \lambda > 0 \).

Solution. Let \( D \) be the closed unit disk. Then

\[
\int_D u(\Delta u + \lambda u) = \int_D 0 = 0.
\]

If we substitute

\[
u \Delta u = \nabla \cdot (u \nabla u) - |\nabla u|^2,
\]

this becomes

\[
\int_D \nabla \cdot (u \nabla u) - \int_D |\nabla u|^2 + \int_D \lambda u^2 = 0.
\]

Applying the Divergence Theorem (in the form

\[
\int_D \nabla \cdot f = \int_{\partial D} f \cdot n
\]

where \( n \) is the outward unit normal) to the first term, we get

\[
\int_{\partial D} uu_n - \int_D |\nabla u|^2 + \int_D \lambda u^2 = 0.
\]

Since \( u_n = -au \) on \( \partial D \), we get

\[
-a \int_{\partial D} u^2 - \int_D |\nabla u|^2 + \lambda \int_D u^2 = 0.
\]

Since \( u \) is not identically zero on \( D \), we have \( \int_D u^2 > 0 \). If \( u \) were constant on \( D \), the equation \( u_n = -au \) on \( \partial D \) would force \( u = 0 \). Thus \( \nabla u \) is not identically zero on \( D \), so \( \int_D |\nabla u|^2 > 0 \).

Finally, \( a \int_{\partial D} u^2 \geq 0 \). Thus solving for \( \lambda \) shows that \( \lambda > 0 \).