Fall 2003 prelim

1A. Show that the differential equation

$$f''(z) = zf(z), \qquad f(0) = 1, \qquad f'(0) = 1$$

has an unique entire solution in the complex plane.

2A. List eight groups of order 36 and prove that they are not isomorphic.

3A. Let A be a 2×2 matrix with complex entries. Prove that the series $I + A + A^2 + \dots$ converges if and only if every eigenvalue of A has absolute value less than 1.

4A. Give an example, with proof, of a nonconstant irreducible polynomial $f(x)$ over $\mathbb{Q}$ with the property that $f(x)$ does not factor into linear factors over the field $K = \mathbb{Q}[x]/(f(x))$.

5A. Let C denote the space of continuous functions on $[0, 1]$. Define

$$d(f, g) = \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|}\, dx.$$

(a) Show that d is a metric on C.
(b) Show that (C, d) is not a complete metric space.

6A. Let $A(m, n)$ be the $m \times n$ matrix with entries

$$a_{ij} = j^i \quad (0 \le i \le m - 1,\ 0 \le j \le n - 1),$$

where $0^0 = 1$ by definition. Regarding the entries of $A(m, n)$ as representing congruence classes (mod p), determine the rank of $A(m, n)$ over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for all $m, n \ge 1$ and all primes p.

7A. Let $D = \{z \in \mathbb{C} : |z| \le 1\} - \{1, -1\}$. Find an explicit continuous function $f : D \to \mathbb{R}$ satisfying all the following conditions:

- f is harmonic on the interior of D (the open unit disk),
- $f(z) = 1$ when $|z| = 1$ and $\mathrm{Im}(z) > 0$, and
- $f(z) = -1$ when $|z| = 1$ and $\mathrm{Im}(z) < 0$.

8A. Let p be a prime, and let G be the group $\mathbb{Z}/p^2\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. How many automorphisms does G have?

9A. Let $f : [0, 1] \to [0, 1]$ be an increasing (not strictly increasing) function such that

$$f\left(\sum_{j=1}^{\infty} a_j 3^{-j}\right) = \sum_{j=1}^{\infty} \frac{a_j}{2} 2^{-j}$$

whenever the a_j are 0 or 2. Prove that there is a constant C_0 such that

$$|f(x) - f(y)| \le C_0 |x - y|^{(\log 2)/(\log 3)}$$

for all $x, y \in [0, 1]$.

1B. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^n+1}\,dx$, where $n \geq 4$ is an even integer.

2B. Let $u_{m,n}$ be an array of numbers for $1 \leq m \leq N$ and $1 \leq n \leq N$. Suppose that $u_{m,n} = 0$ when m is 1 or N, or when n is 1 or N. Suppose also that

$$u_{m,n} = \frac{1}{4}\left(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1}\right)$$

whenever $1 < m < N$ and $1 < n < N$. Show that all the $u_{m,n}$ are zero.

3B. Let A and B be $n \times n$ complex unitary matrices. Prove that $|\det(A+B)| \leq 2^n$.

4B. Let L be a line in $\mathbb{C}$, and let f be an entire function such that $f(\mathbb{C}) \cap L = \emptyset$. Prove that f is constant. (Do not use the theorem of Picard that the image of a nonconstant entire function omits at most one complex number.)

5B. Let n be a positive integer. Let $\phi(n)$ be the Euler phi function, so $\phi(n) = \#(\mathbb{Z}/n\mathbb{Z})^*$. Prove that if $\gcd(n, \phi(n)) > 1$, then there exists a noncyclic group of order n.

6B. Let $f(z)$ be a meromorphic function on the complex plane. Suppose that for every polynomial $p(z) \in \mathbb{C}[z]$ and every closed contour Γ avoiding the poles of f, we have

$$\int_\Gamma p(z)^2 f(z)\,dz = 0.$$

Prove that $f(z)$ is entire.

7B. (a) Let G be a finite group and let X be the set of pairs of commuting elements of G:

$$X = \{(g,h) \in G \times G : gh = hg\}.$$

Prove that $|X| = c|G|$ where c is the number of conjugacy classes in G.

(b) Compute the number of pairs of commuting permutations on five letters.

8B. The set of 5×5 complex matrices A satisfying $A^3 = A^2$ is a union of conjugacy classes. How many conjugacy classes?

9B. Let $\lambda, a \in \mathbb{R}$, with $a > 0$. Let $u(x,y)$ be an infinitely differentiable function defined on an open neighborhood of $x^2 + y^2 \leq 1$ such that

$$\begin{aligned} \Delta u + \lambda u &= 0 \qquad && \text{in } x^2 + y^2 < 1 \\ u_n &= -au && \text{on } x^2 + y^2 = 1. \end{aligned}$$

Here Δ is the Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2$, and u_n denotes the directional derivative of u in the direction of the outward unit normal (pointing away from the origin). Prove that if u is not identically zero in $x^2 + y^2 < 1$, then $\lambda > 0$.