Problem 1  1. Show that a real $2 \times 2$ matrix $A$ satisfies $A^2 = -I$ if and only if

$$A = \begin{pmatrix} \pm \sqrt{pq-1} & -p \\ q & \mp \sqrt{pq - 1} \end{pmatrix}$$

where $p$ and $q$ are real numbers such that $pq \geq 1$ and both upper or both lower signs should be chosen in the double signs.

2. Show that there is no real $2 \times 2$ matrix $A$ such that

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 - \varepsilon \end{pmatrix}$$

with $\varepsilon > 0$.

Problem 2  1. For $0 \leq \theta \leq \frac{\pi}{2}$, show that

$$\sin \theta \geq \frac{2}{\pi} \theta.$$  

2. By using Part 1, or by any other method, show that if $\lambda < 1$, then

$$\lim_{R \to \infty} R^\lambda \int_0^{\frac{\pi}{2}} e^{-R \sin \theta} d\theta = 0.$$  

Problem 3 Let $A$ be a nonsingular real $n \times n$ matrix. Prove that there exists a unique orthogonal matrix $Q$ and a unique positive definite symmetric matrix $B$ such that $A = QB$.

Problem 4 Let $G$ be a group of order 120, let $H$ be a subgroup of order 24, and assume that there is at least one left coset of $H$ (other than $H$ itself) which is equal to some right coset of $H$. Prove that $H$ is a normal subgroup of $G$.

Problem 5 By the Fundamental Theorem of Algebra, the polynomial $x^3 + 2x^2 + 7x + 1$ has three complex roots, $\alpha_1$, $\alpha_2$, and $\alpha_3$. Compute $\alpha_1^3 + \alpha_2^3 + \alpha_3^3$. 
Problem 6 Evaluate the integral
\[ \int_0^\infty \frac{x^{a-1}}{x+1} \, dx \]
where \( a \) is a complex number. What restrictions must be put on \( a \)?

Problem 7 Let
\[ f(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2} \, dt \]
for \( x > 0 \).

1. Show that \( 0 < f(x) < \frac{1}{x} \).
2. Show that \( f(x) \) is strictly decreasing for \( x > 0 \).

Problem 8 Let \( f \) be a real valued continuous function on a compact interval \([a, b]\). Given \( \varepsilon > 0 \), show that there is a polynomial \( p \) such that \( p(a) = f(a) \), \( p'(a) = 0 \), and \( |p(x) - f(x)| < \varepsilon \) for \( x \in [a, b] \).

Problem 9 Let \( u(x) \), \( 0 \leq x \leq 1 \), be a real valued \( C^2 \) function which satisfies the differential equation
\[ u''(x) = e^x u(x) \].

1. Show that if \( 0 < x_0 < 1 \), then \( u \) cannot have a positive local maximum at \( x_0 \). Similarly, show that \( u \) cannot have a negative local minimum at \( x_0 \).
2. Now suppose that \( u(0) = u(1) = 0 \). Prove that \( u(x) \equiv 0 \), \( 0 \leq x \leq 1 \).

Problem 10 Prove that for each \( \lambda > 1 \) the equation \( z = \lambda - e^{-z} \) in the half-plane \( \Re z \geq 0 \) has exactly one root, and that this root is real.

Problem 11 1. Let \( G \) be a cyclic group, and let \( a, b \in G \) be elements which are not squares. Prove that \( ab \) is a square.
2. Give an example to show that this result is false if the group is not cyclic.

Problem 12 Let \( A \) be an \( n \times n \) real matrix and \( A^t \) its transpose. Show that \( A^t A \) and \( A^t \) have the same range.
Problem 13 Let $P(z)$ be a polynomial of degree $< k$ with complex coefficients. Let $\omega_1, \ldots, \omega_k$ be the $k$th roots of unity in $\mathbb{C}$. Prove that

$$\frac{1}{k} \sum_{i=1}^{k} P(\omega_i) = P(0).$$

Problem 14 Let $M_n(F)$ denote the ring of $n \times n$ matrices over a field $F$. For $n \geq 1$, does there exist a ring homomorphism from $M_{n+1}(F)$ onto $M_n(F)$?

Problem 15 For each $k > 0$, let $X_k$ be the set of analytic functions $f(z)$ on the open unit disc $D$ such that

$$\sup_{z \in D} \left\{(1 - |z|)k |f(z)|\right\}$$

is finite. Show that $f \in X_k$ if and only if $f' \in X_{k+1}$.

Problem 16 A function $f : [0, 1] \to \mathbb{R}$ is said to be upper semicontinuous if given $x \in [0, 1]$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|y - x| < \delta$, then $f(y) < f(x) + \varepsilon$. Prove that an upper semicontinuous function $f$ on $[0, 1]$ is bounded above and attains its maximum value at some point $p \in [0, 1]$.

Problem 17 Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where all the $a_n$ are nonnegative reals, and the series has radius of convergence 1. Prove that $f(z)$ cannot be analytically continued to a function analytic in a neighborhood of $z = 1$.

Problem 18 Solve the differential equations

$$\frac{dy_1}{dx} = -3y_1 + 10y_2,$$

$$\frac{dy_2}{dx} = -3y_1 + 8y_2.$$

Problem 19 Let $A_1 \geq A_2 \geq \cdots \geq A_k \geq 0$. Evaluate

$$\lim_{n \to \infty} (A_1^n + A_2^n + \cdots + A_k^n)^{1/n}.$$ 

Note: See also Problem ??.

Problem 20 Let $F$ be a field of characteristic $p > 0$, $p \neq 3$. If $\alpha$ is a zero of the polynomial $f(x) = x^p - x + 3$ in an extension field of $F$, show that $f(x)$ has $p$ distinct zeros in the field $F(\alpha)$. 

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