Problem 1 Show that if a subgroup \(H\) of a group \(G\) has just one left coset different from itself, then it is a normal subgroup of \(G\).

Problem 2 Let \(\mathbb{Z}\) be the ring of integers and \(\mathbb{Z}[x]\) the polynomial ring over \(\mathbb{Z}\). Show that 
\[x^6 + 539x^5 - 511x + 847\]
is irreducible in \(\mathbb{Z}[x]\).

Problem 3 Let \(f : \mathbb{R}^m \rightarrow \mathbb{R}^n, n \geq 2\), be a linear transformation of rank \(n-1\). Let \(f(v) = (f_1(v), f_2(v), \ldots, f_n(v))\) for \(v \in \mathbb{R}^m\). Show that a necessary and sufficient condition for the system of inequalities \(f_i(v) > 0, i = 1, \ldots, n\), to have no solution is that there exist real numbers \(\lambda_i \geq 0\), not all zero, such that 
\[
\sum_{i=1}^{n} \lambda_i f_i = 0.
\]

Problem 4 Let 
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
be a real matrix with \(a, b, c, d > 0\). Show that \(A\) has an eigenvector 
\[
\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2
\]
with \(x, y > 0\).

Problem 5 1. Show that there is a unique analytic branch outside the unit circle of the function \(f(z) = \sqrt{z^2 + z + 1}\) such that \(f(t)\) is positive when \(t > 1\).

2. Using the branch determined in Part 1, calculate the integral 
\[
\frac{1}{2\pi i} \int_{C_r} \frac{dz}{\sqrt{z^2 + z + 1}}
\]
where \(C_r\) is the positively oriented circle \(|z| = r\) and \(r > 1\).
Problem 6 Let $\rho > 0$. Show that for $n$ large enough, all the zeros of

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$$

lie in the circle $|z| < \rho$.

Problem 7 Let $f : \mathbb{R} \to \mathbb{R}$ be $C^1$ and let

$$u = f(x)$$
$$v = -y + xf(x).$$

If $f'(x_0) \neq 0$, show that this transformation is locally invertible near $(x_0, y_0)$ and the inverse has the form

$$x = g(u)$$
$$y = -v + ug(u).$$

Problem 8 Let $\varphi(s)$ be a $C^2$ function on $[1, 2]$ with $\varphi$ and $\varphi'$ vanishing at $s = 1, 2$. Prove that there is a constant $C > 0$ such that for any $\lambda > 1$,

$$\left| \int_1^2 e^{i\lambda x} \varphi(x) \, dx \right| \leq C \frac{\lambda}{\lambda^2}.$$

Problem 9 Consider the solution curve $(x(t), y(t))$ to the equations

$$\frac{dx}{dt} = 1 + \frac{1}{2}x^2 \sin y$$
$$\frac{dy}{dt} = 3 - x^2$$

with initial conditions $x(0) = 0$ and $y(0) = 0$. Prove that the solution must cross the line $x = 1$ in the $xy$ plane by the time $t = 2$.

Problem 10 Let $C^{1/3}$ be the set of real valued functions $f$ on the closed interval $[0, 1]$ such that

1. $f(0) = 0$;

2. $\|f\|$ is finite, where by definition

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1/3}} \mid x \neq y \right\}.$$
Verify that $\| \cdot \|$ is a norm for the space $C^{1/3}$, and prove that $C^{1/3}$ is complete with respect to this norm.

**Problem 11** Let $S_n$ denote the group of permutations of $n$ objects. Find four different subgroups of $S_4$ isomorphic to $S_3$ and nine isomorphic to $S_2$.

**Problem 12** Let $F_q$ be a finite field with $q$ elements and let $V$ be an $n$-dimensional vector space over $F_q$.

1. Determine the number of elements in $V$.
2. Let $GL_n(F_q)$ denote the group of all $n \times n$ nonsingular matrices $A$ over $F_q$. Determine the order of $GL_n(F_q)$.
3. Let $SL_n(F_q)$ denote the subgroup of $GL_n(F_q)$ consisting of matrices with determinant 1. Find the order of $SL_n(F_q)$.

**Problem 13** Let $A$ be a $2 \times 2$ matrix over $\mathbb{C}$ which is not a scalar multiple of the identity matrix $I$. Show that any $2 \times 2$ matrix $X$ over $\mathbb{C}$ commuting with $A$ has the form $X = \alpha I + \beta A$, where $\alpha, \beta \in \mathbb{C}$.

**Problem 14** Suppose $V$ is an $n$-dimensional vector space over the field $\mathbb{F}$. Let $W \subset V$ be a subspace of dimension $r < n$. Show that

$$W = \bigcap \{ U \mid U \text{ is an } (n-1) \text{ - dimensional subspace of } V \text{ and } W \subset U \}. $$

**Problem 15** Let $\mathbb{Z}_3$ be the field of integers mod 3 and $\mathbb{Z}_3[x]$ the corresponding polynomial ring. Decompose $x^3 + x + 2$ into irreducible factors in $\mathbb{Z}_3[x]$.

**Problem 16** Let $p(z)$ be a nonconstant polynomial with real coefficients such that for some real number $a$, $p(a) \neq 0$ but $p'(a) = p''(a) = 0$. Prove that the equation $p(z) = 0$ has a nonreal root.

**Problem 17** Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$. Show that

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

is entire and that for $0 < r < R$, there is a constant $M$ such that

$$|h(z)| \leq Me^{\frac{|z|}{r}}.$$
Problem 18 Show there is a unique continuous real valued function \( f : [0, 1] \rightarrow \mathbb{R} \) such that

\[
f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}} \, dy.
\]

Problem 19 Let \( x(t) \) be the solution of the differential equation

\[
x''(t) + 8x'(t) + 25x(t) = 2 \cos t
\]

with initial conditions \( x(0) = 0 \) and \( x'(0) = 0 \). Show that for suitable constants \( \alpha \) and \( \delta \),

\[
\lim_{t \to \infty} (x(t) - \alpha \cos(t - \delta)) = 0.
\]

Problem 20 Evaluate

\[
\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 20} \, dx.
\]