Preliminary Exam - Summer 1984

Problem 1 *Show that if a subgroup H of a group G has just one left coset different from itself, then it is a normal subgroup of G.*

Problem 2 *Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}[x]$ the polynomial ring over $\mathbb{Z}$. Show that*

$$x^6 + 539x^5 - 511x + 847$$

is irreducible in $\mathbb{Z}[x]$.

Problem 3 *Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $n \geqslant 2$, be a linear transformation of rank $n-1$. Let $f(v) = (f_1(v), f_2(v), \ldots, f_n(v))$ for $v \in \mathbb{R}^m$. Show that a necessary and sufficient condition for the system of inequalities $f_i(v) > 0$, $i = 1, \ldots, n$, to have no solution is that there exist real numbers $\lambda_i \geqslant 0$, not all zero, such that*

$$\sum_{i=1}^{n} \lambda_i f_i = 0.$$

Problem 4 *Let*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a real matrix with $a, b, c, d > 0$. Show that A has an eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

with $x, y > 0$.

Problem 5

1. *Show that there is a unique analytic branch outside the unit circle of the function $f(z) = \sqrt{z^2 + z + 1}$ such that $f(t)$ is positive when $t > 1$.*

2. *Using the branch determined in Part 1, calculate the integral*

$$\frac{1}{2\pi i} \int_{C_r} \frac{dz}{\sqrt{z^2 + z + 1}}$$

where C_r is the positively oriented circle $|z| = r$ and $r > 1$.

Problem 6 *Let $\rho > 0$. Show that for n large enough, all the zeros of*

$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}$$

lie in the circle $|z| < \rho$.

Problem 7 *Let $f : \mathbb{R} \to \mathbb{R}$ be C^1 and let*

$$\begin{aligned} u &= f(x) \\ v &= -y + xf(x). \end{aligned}$$

If $f'(x_0) \neq 0$, show that this transformation is locally invertible near (x_0, y_0) and the inverse has the form

$$\begin{aligned} x &= g(u) \\ y &= -v + ug(u). \end{aligned}$$

Problem 8 *Let $\varphi(s)$ be a C^2 function on $[1, 2]$ with φ and φ' vanishing at $s = 1, 2$. Prove that there is a constant $C > 0$ such that for any $\lambda > 1$,*

$$\left| \int_1^2 e^{i\lambda x} \varphi(x)\, dx \right| \leqslant \frac{C}{\lambda^2}.$$

Problem 9 *Consider the solution curve $(x(t), y(t))$ to the equations*

$$\begin{aligned} \frac{dx}{dt} &= 1 + \frac{1}{2}x^2 \sin y \\ \frac{dy}{dt} &= 3 - x^2 \end{aligned}$$

with initial conditions $x(0) = 0$ and $y(0) = 0$. Prove that the solution must cross the line $x = 1$ in the xy plane by the time $t = 2$.

Problem 10 *Let $C^{1/3}$ be the set of real valued functions f on the closed interval $[0, 1]$ such that*

1. $f(0) = 0$;
2. $\|f\|$ *is finite, where by definition*

$$\|f\| = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{1/3}} \mid x \neq y \right\}.$$

Verify that $\|\cdot\|$ is a norm for the space $C^{1/3}$, and prove that $C^{1/3}$ is complete with respect to this norm.

Problem 11 *Let S_n denote the group of permutations of n objects. Find four different subgroups of S_4 isomorphic to S_3 and nine isomorphic to S_2.*

Problem 12 *Let $\mathbf{F}_q$ be a finite field with q elements and let V be an n-dimensional vector space over $\mathbf{F}_q$.*

1. *Determine the number of elements in V.*
2. *Let $GL_n(\mathbf{F}_q)$ denote the group of all $n \times n$ nonsingular matrices A over $\mathbf{F}_q$. Determine the order of $GL_n(\mathbf{F}_q)$.*
3. *Let $SL_n(\mathbf{F}_q)$ denote the subgroup of $GL_n(\mathbf{F}_q)$ consisting of matrices with determinant 1. Find the order of $SL_n(\mathbf{F}_q)$.*

Problem 13 *Let A be a 2×2 matrix over $\mathbb{C}$ which is not a scalar multiple of the identity matrix I. Show that any 2×2 matrix X over $\mathbb{C}$ commuting with A has the form $X = \alpha I + \beta A$, where $\alpha, \beta \in \mathbb{C}$.*

Problem 14 *Suppose V is an n-dimensional vector space over the field $\mathbf{F}$. Let $W \subset V$ be a subspace of dimension $r < n$. Show that*

$$W = \bigcap \{U \mid U \, is \, an \, (n-1)-dimensional \; subspace \; of \; V \, and \, W \subset U\}.$$

Problem 15 *Let $\mathbb{Z}_3$ be the field of integers $\bmod 3$ and $\mathbb{Z}_3[x]$ the corresponding polynomial ring. Decompose $x^3 + x + 2$ into irreducible factors in $\mathbb{Z}_3[x]$.*

Problem 16 *Let $p(z)$ be a nonconstant polynomial with real coefficients such that for some real number a, $p(a) \neq 0$ but $p'(a) = p''(a) = 0$. Prove that the equation $p(z) = 0$ has a nonreal root.*

Problem 17 *Suppose*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

has radius of convergence $R > 0$. Show that

$$h(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!}$$

is entire and that for $0 < r < R$, there is a constant M such that

$$|h(z)| \leqslant M e^{|z|/r}.$$

Problem 18 *Show there is a unique continuous real valued function* $f : [0,1] \to \mathbb{R}$ *such that*

$$f(x) = \sin x + \int_0^1 \frac{f(y)}{e^{x+y+1}}\, dy.$$

Problem 19 *Let* $x(t)$ *be the solution of the differential equation*

$$x''(t) + 8x'(t) + 25x(t) = 2\cos t$$

with initial conditions $x(0) = 0$ *and* $x'(0) = 0$. *Show that for suitable constants* α *and* δ,

$$\lim_{t\to\infty} (x(t) - \alpha\cos(t-\delta)) = 0.$$

Problem 20 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4x + 20}\, dx\,.$$