Preliminary Exam - Summer 1983

Problem 1 *The number* 21982145917308330487013369 *is the thirteenth power of a positive integer. Which positive integer?*

Problem 2 *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function such that*

$$\left(1 + |z|^k\right)^{-1} \frac{d^m f}{dz^m}$$

is bounded for some k and m. Prove that $d^n f/dz^n$ is identically zero for sufficiently large n. How large must n be, in terms of k and m?

Problem 3 *Let A be an $n \times n$ complex matrix, and let χ and μ be the characteristic and minimal polynomials of A. Suppose that*

$$\chi(x) = \mu(x)(x - i),$$

$$\mu(x)^2 = \chi(x)(x^2 + 1).$$

Determine the Jordan Canonical Form of A.

Problem 4 *Outline a proof, starting from basic properties of the real numbers, of the following theorem: Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f'(x) = 0$ for all $x \in (a, b)$. Then $f(b) = f(a)$.*

Problem 5 *Let $b_1, b_2, \ldots$ be positive real numbers with*

$$\lim_{n\to\infty} b_n = \infty \text{ and } \lim_{n\to\infty} (b_n/b_{n+1}) = 1.$$

Assume also that $b_1 < b_2 < b_3 < \cdots$. Show that the set of quotients $(b_m/b_n)_{1 \leqslant n < m}$ is dense in $(1, \infty)$.

Problem 6 *Let V be a real vector space of dimension n with a positive definite inner product. We say that two bases (a_i) and (b_i) have the same orientation if the matrix of the change of basis from (a_i) to (b_i) has a positive determinant. Suppose now that (a_i) and (b_i) are orthonormal bases with the same orientation. Show that $(a_i + 2b_i)$ is again a basis of V with the same orientation as (a_i).*

Problem 7 *Compute*

$$\int_0^\infty \frac{\log x}{x^2+a^2}\,dx$$

where $a > 0$ is a constant.

Problem 8 *Let G_1, G_2, and G_3 be finite groups, each of which is generated by its commutators (elements of the form $xyx^{-1}y^{-1}$). Let A be a subgroup of $G_1 \times G_2 \times G_3$, which maps surjectively, by the natural projection map, to the partial products $G_1 \times G_2$, $G_1 \times G_3$ and $G_2 \times G_3$. Show that A is equal to $G_1 \times G_2 \times G_3$.*

Problem 9 *Suppose Ω is a bounded domain in $\mathbb{C}$ with a boundary consisting of a smooth Jordan curve γ. Let f be holomorphic on a neighborhood of the closure of Ω, and suppose that $f(z) \neq 0$ for $z \in \gamma$. Let $z_1, \ldots, z_k$ be the zeros of f in Ω, and let n_j be the order of the zero of f at z_j (for $j = 1, \ldots, k$).*

1. *Use Cauchy's integral formula to show that*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{j=1}^{k} n_j.$$

2. *Suppose that f has only one zero z_1 in Ω with multiplicity $n_1 = 1$. Find a boundary integral involving f whose value is the point z_1.*

Problem 10 *Let f be a twice differentiable real valued function on $[0, 2\pi]$, with $\int_0^{2\pi} f(x)dx = 0 = f(2\pi) - f(0)$. Show that*

$$\int_0^{2\pi} (f(x))^2\,dx \leqslant \int_0^{2\pi} (f'(x))^2\,dx\,.$$

Problem 11 *Find the eigenvalues, eigenvectors, and the Jordan Canonical Form of*

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

considered as a matrix with entries in $\mathbf{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

Problem 12 *Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ which maps open sets to open sets must be monotonic.*

Problem 13 *Let A be an $n{\times}n$ complex matrix, all of whose eigenvalues are equal to 1. Suppose that the set $\{A^n \mid n = 1, 2, \ldots\}$ is bounded. Show that A is the identity matrix.*

Problem 14 *Let G be a transitive subgroup of the group S_n of permutations of n objects $\{1, \ldots, n\}$. Suppose that G is a simple group and that $\sim$ is an equivalence relation on $\{1, \ldots, n\}$ such that $i \sim j$ implies that $\sigma(i) \sim \sigma(j)$ for all $\sigma \in G$. What can one conclude about the relation $\sim$?*

Problem 15 *Let f be analytic on and inside the unit circle$C = \{z \mid |z| = 1\}$. Let L be the length of the image of C under f. Show that $L \geqslant 2\pi|f'(0)|$.*

Problem 16 *Let Ω be an open subset of $\mathbb{R}^2$, and let $f : \Omega \to \mathbb{R}^2$ be a smooth map. Assume that f preserves orientation and maps any pair of orthogonal curves to a pair of orthogonal curves. Show that f is holomorphic.*
Note: Here we identify $\mathbb{R}^2$ with $\mathbb{C}$.

Problem 17 *Let A be an $n \times n$ Hermitian matrix satisfying the condition*

$$A^5 + A^3 + A = 3I\,.$$

Show that $A = I$.

Problem 18 *Find all real valued C^1 solutions $y(x)$ of the differential equation*

$$x\frac{dy}{dx} + y = x \quad (-1 < x < 1).$$

Problem 19 *Compute the area of the image of the unit disc $\{z|\ |z|<1\}$ under the map $f(z) = z + z^2/2$.*

Problem 20 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, periodic of period 1, and nonnegative. Show that*

$$\frac{d}{dx}\left(\frac{f(x)}{1 + cf(x)}\right) \to 0 \quad (as\ c \to \infty)$$

uniformly in x.