Preliminary Exam - Summer 1982

Problem 1 *Determine the Jordan Canonical Form of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Problem 2 *Compute the integral*

$$\int_0^\infty \frac{x^{50}}{x^{100}+1}\, dx\,.$$

Problem 3 *Let K be a nonempty compact set in a metric space with distance function d. Suppose that $\varphi\colon K \to K$ satisfies*

$$d(\varphi(x), \varphi(y)) < d(x, y)$$

for all $x \neq y$ in K. Show there exists precisely one point $x \in K$ such that$x = \varphi(x)$.

Problem 4 *Let G be a group with generators a and b satisfying*

$$a^{-1}b^2a = b^3, \qquad b^{-1}a^2b = a^3.$$

Is G trivial?

Problem 5 *Let $0 < a_0 \leqslant a_1 \leqslant \cdots \leqslant a_n$. Prove that the equation*

$$a_0z^n + a_1z^{n-1} + \cdots + a_n = 0$$

has no roots in the disc $|z| < 1$.

Problem 6 *Suppose f is a differentiable real valued function such that $f'(x) > f(x)$ for all $x \in \mathbb{R}$ and $f(0) = 0$. Prove that $f(x) > 0$ for all positive x.*

Problem 7 *Let V be the vector space of all real 3×3 matrices and let A be the diagonal matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate the determinant of the linear transformation T on V defined by $T(X) = \frac{1}{2}(AX + XA)$.

Problem 8 *Let n be a positive integer.*

1. *Show that the binomial coefficient*

   $$c_n = \binom{2n}{n}$$

   is even.

2. *Prove that c_n is divisible by 4 if and only if n is not a power of 2.*

Problem 9 *Determine the complex numbers z for which the power series*

$$\sum_{n=1}^{\infty} \frac{z^n}{n^{\log n}}$$

and its term by term derivatives of all orders converge absolutely.

Problem 10 *For complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, prove*

$$\limsup_n \left| \sum_{j=1}^{k} \alpha_j^n \right|^{1/n} = \sup_j |\alpha_j|.$$

Note: See also Problem **??**.

Problem 11 *Let $s(y)$ and $t(y)$ be real differentiable functions of y, $-\infty < y < \infty$, such that the complex function*

$$f(x + iy) = e^x \left(s(y) + it(y)\right)$$

is complex analytic with $s(0) = 1$ and $t(0) = 0$. Determine $s(y)$ and $t(y)$.

Problem 12 *Determine (with proofs) which of the following polynomials are irreducible over the field* $\mathbb{Q}$ *of rationals.*

1. $x^2 + 3$
2. $x^2 - 169$
3. $x^3 + x^2 + x + 1$
4. $x^3 + 2x^2 + 3x + 4.$

Problem 13 *Let* $f : [0, \pi] \to \mathbb{R}$ *be continuous and such that*

$$\int_0^\pi f(x) \sin nx \, dx = 0$$

for all integers $n \geqslant 1$. *Is* f *identically* 0*?*

Problem 14 *Let* A *be a real* $n \times n$ *matrix such that* $\langle Ax, x \rangle \geqslant 0$ *for every real* n*-vector* x. *Show that* $Au = 0$ *if and only if* $A^t u = 0$.

Problem 15 *Let* $f(z)$ *be analytic on the open unit disc* $\mathbb{D}=\{z||z|<1\}$. *Prove that there is a sequence* (z_n) *in* $\mathbb{D}$ *such that* $|z_n| \to 1$ *and* $(f(z_n))$ *is bounded.*

Problem 16 *A square matrix* A *is nilpotent if* $A^k = 0$ *for some positive integer* k.

1. *If* A *and* B *are nilpotent, is* $A+B$ *nilpotent? Proof or counterexample.*
2. *Prove: If* A *is nilpotent, then* $I - A$ *is invertible.*

Problem 17 *Let* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *and assume that* 0 *is a* regular value *of* f *(i.e., the differential of* f *has rank* 2 *at each point of* $f^{-1}(0)$*). Prove that* $\mathbb{R}^3 \setminus f^{-1}(0)$ *is arcwise connected.*

Problem 18 *Let* E *be the set of all continuous real valued functions* $u : [0, 1] \to \mathbb{R}$ *satisfying*

$$|u(x) - u(y)| \leqslant |x - y|, \quad 0 \leqslant x, y \leqslant 1, \quad u(0) = 0.$$

Let $\varphi : E \to \mathbb{R}$ *be defined by*

$$\varphi(u) = \int_0^1 \left(u(x)^2 - u(x) \right) \, dx \, .$$

Show that φ *achieves its maximum value at some element of* E.

Problem 19 *Let V be a finite-dimensional vector space over the rationals $\mathbb{Q}$ and let M be an automorphism of V such that M fixes no nonzero vector in V. Suppose that M^p is the identity map on V, where p is a prime number. Show that the dimension of V is divisible by $p-1$.*

Problem 20 *Let $M_{2\times 2}$ be the four-dimensional vector space of all 2×2 real matrices and define $f : M_{2\times 2} \to M_{2\times 2}$ by $f(X) = X^2$.*

1. *Show that f has a local inverse near the point*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2. *Show that f does not have a local inverse near the point*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$