Preliminary Exam - Summer 1980

Problem 1 *Exhibit a real 3×3 matrix having minimal polynomial* $(t^2+1)(t-10)$*, which, as a linear transformation of* $\mathbb{R}^3$*, leaves invariant the line* L *through* $(0,0,0)$ *and* $(1,1,1)$ *and the plane through* $(0,0,0)$ *perpendicular to* L.

Problem 2 *Which of the following matrix equations have a real matrix solution* X*? (It is not necessary to exhibit solutions.)*

1.

$$X^3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{pmatrix},$$

2.

$$2X^5 + X = \begin{pmatrix} 3 & 5 & 0 \\ 5 & 1 & 9 \\ 0 & 9 & 0 \end{pmatrix},$$

3.

$$X^6 + 2X^4 + 10X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

4.

$$X^4 = \begin{pmatrix} 3 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Problem 3 *Let* $T : V \to V$ *be an invertible linear transformation of a vector space* V*. Denote by* G *the group of all maps* $f_{k,a} : V \to V$ *where* $k \in \mathbb{Z}$*,* $a \in V$*, and for* $x \in V$,

$$f_{k,a}(x) = T^k x + a \quad (x \in V).$$

Prove that the commutator subgroup G' *of* G *is isomorphic to the additive group of the vector space* $(T-I)V$*, the image of* $T-I$*. (*G' *is generated by all* $ghg^{-1}h^{-1}$*,* g *and* h *in* G*.)*

Problem 4 *Let G be a finite group and $H \subset G$ a subgroup.*

1. *Show that the number of subgroups of G of the form xHx^{-1} for some $x \in G$ is $\leqslant$ the index of H in G.*

2. *Prove that some element of G is not in any subgroup of the form xHx^{-1}, $x \in G$.*

Problem 5 *Consider the differential equations*

$$\frac{dx}{dt} = -x + y, \qquad \frac{dy}{dt} = \log(20 + x) - y.$$

Let $x(t)$ and $y(t)$ be a solution defined for all $t \geqslant 0$ with $x(0) > 0$ and $y(0) > 0$. Prove that $x(t)$ and $y(t)$ are bounded.

Problem 6 *Let C denote the positively oriented circle $|z| = 2, z \in \mathbb{C}$. Evaluate the integral*

$$\int_C \sqrt{z^2 - 1}\, dz$$

where the branch of the square root is chosen so that $\sqrt{2^2 - 1} > 0$.

Problem 7 *Exhibit a conformal map from $\{z \in \mathbb{C} \mid |z| < 1, \Re z > 0\}$ onto $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.*

Problem 8 *Give an example of a subset of $\mathbb{R}$ having uncountably many connected components. Can such a subset be open? Closed?*

Problem 9 *For each $(a, b, c) \in \mathbb{R}^3$, consider the series*

$$\sum_{n=3}^{\infty} \frac{a^n}{n^b (\log n)^c}.$$

Determine the values of (a, b, c) for which the series

1. *converges absolutely;*

2. *converges but not absolutely;*

3. *diverges.*

Problem 10 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function whose partial derivatives of order $\leqslant 2$ are everywhere defined and continuous.*

1. *Let $a \in \mathbb{R}^n$ be a critical point of f (i.e., $\frac{\partial f}{\partial x_j}(a) = 0,\ i = 1, \ldots, n$). Prove that a is a local minimum provided the Hessian matrix*

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

is positive definite at $x = a$.

2. *Assume the Hessian matrix is positive definite at all x. Prove that f has, at most, one critical point.*

Problem 11 *Prove that every finite group is isomorphic to*

1. *A group of permutations;*

2. *A group of even permutations.*

Problem 12 *Let $S \subset \mathbb{R}^3$ denote the ellipsoidal surface defined by*

$$2x^2 + (y-1)^2 + (z-10)^2 = 1.$$

Let $T \subset \mathbb{R}^3$ be the surface defined by

$$z = \frac{1}{x^2 + y^2 + 1}.$$

Prove that there exist points $p \in S,\ q \in T$, such that the line $\overline{pq}$ is perpendicular to S at p and to T at q.

Problem 13 *Let $\mathfrak{J}$ be the ideal in the ring $\mathbb{Z}[x]$ (of polynomials with integer coefficients) generated by $x - 5$ and 14. Find $n \in \mathbb{Z}$ such that $0 \leqslant n \leqslant 13$ and $(x^3 + 2x + 1)^{50} - n \in \mathfrak{J}$.*

Problem 14 *Let A and B be real 2×2 matrices such that$A^2 = B^2 = I$ and $AB + BA = 0$. Prove there exists a real nonsingular matrix T with*

$$TAT^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad TBT^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Problem 15 *Let E be a finite-dimensional vector space over a field $\mathbf{F}$. Suppose $B : E \times E \to \mathbf{F}$ is a bilinear map (not necessarily symmetric). Define subspaces*

$$E_1 = \{x \in E \mid B(x,y) = 0 \ for \ \ all \ y \in E\},$$
$$E_2 = \{y \in E \mid B(x,y) = 0 \ for \ \ all \ x \in E\}$$

Prove that $\dim E_1 = \dim E_2$.

Problem 16 *Let (a_n) be a sequence of nonzero real numbers. Prove that the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$*

$$f_n(x) = \frac{1}{a_n} \sin(a_n x) + \cos(x + a_n)$$

has a subsequence converging to a continuous function.

Problem 17 *Let $f : \mathbb{R} \to \mathbb{R}$ be monotonically increasing (perhaps discontinuous). Suppose $0 < f(0)$ and $f(100) < 100$. Prove $f(x) = x$ for some x.*

Problem 18 *How many zeros does the complex polynomial*

$$3z^9 + 8z^6 + z^5 + 2z^3 + 1$$

have in the annulus $1 < |z| < 2$?

Problem 19 *Let f be a meromorphic function on $\mathbb{C}$ which is analytic in a neighborhood of 0. Let its Maclaurin series be*

$$\sum_{k=0}^{\infty} a_k z^k$$

with all $a_k \geqslant 0$. Suppose there is a pole of modulus $r > 0$ and no pole has modulus $< r$. Prove there is a pole at $z = r$.

Problem 20 *Prove that the initial value problem*

$$\frac{dx}{dt} = 3x + 85 \cos x, \qquad x(0) = 77$$

has a solution $x(t)$ defined for all $t \in \mathbb{R}$.