Problem 1 Prove that the matrix
\[
\begin{pmatrix}
0 & 5 & 1 & 0 \\
5 & 0 & 5 & 0 \\
1 & 5 & 0 & 5 \\
0 & 0 & 5 & 0
\end{pmatrix}
\]
has two positive and two negative eigenvalues (counting multiplicities).

Problem 2 Let \( F \) be a subfield of a field \( K \). Let \( p \) and \( q \) be polynomials over \( F \). Prove that their greatest common divisor in the ring of polynomials over \( F \) is the same as their \( \gcd \) in the ring of polynomials over \( K \).

Problem 3 Let \( X \) be the space of orthogonal real \( n \times n \) matrices. Let \( v_0 \in \mathbb{R}^n \). Locate and describe the elements of \( X \), where the map
\[
f : X \to \mathbb{R}, \quad f(A) = \langle v_0, Av_0 \rangle
\]
takes its maximum and minimum values.

Problem 4 Prove that the group of automorphisms of a cyclic group of prime order \( p \) is cyclic and find its order.

Problem 5 1. Give an example of a differentiable function \( f : \mathbb{R} \to \mathbb{R} \)
whose derivative \( f' \) is not continuous.

2. Let \( f \) be as in Part 1. If \( f'(0) < 2 < f'(1) \), prove that \( f'(x) = 2 \) for some \( x \in [0, 1] \).

Problem 6 Let \( f(z) = a_0 + a_1 z + \cdots + a_n z^n \) be a complex polynomial of degree \( n > 0 \). Prove
\[
\frac{1}{2\pi i} \int_{|z|=R} z^{n-1} |f(z)|^2 \, dz = a_0 \bar{a}_n R^{2n}.
\]
Problem 7 Let $f$ be a continuous complex valued function on $[0, 1]$, and define the function $g$ by

$$g(z) = \int_0^1 f(t)e^{tz} dt \quad (z \in \mathbb{C}).$$

Prove that $g$ is analytic in the entire complex plane.

Problem 8 Let $U \subset \mathbb{R}^n$ be an open set. Suppose that the map $h : U \to \mathbb{R}^n$ is a homeomorphism from $U$ onto $\mathbb{R}^n$, which is uniformly continuous. Prove $U = \mathbb{R}^n$.

Problem 9 Prove that a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ has

1. a one-dimensional invariant subspace, and
2. a two-dimensional invariant subspace.

Problem 10 Find all pairs of $C^\infty$ functions $x(t)$ and $y(t)$ on $\mathbb{R}$ satisfying

$$x'(t) = 2x(t) - y(t), \quad y'(t) = x(t).$$

Problem 11 Let $A$ and $B$ be $n \times n$ matrices over a field $F$ such that $A^2 = A$ and $B^2 = B$. Suppose that $A$ and $B$ have the same rank. Prove that $A$ and $B$ are similar.

Problem 12 Which rational numbers $t$ are such that

$$3t^3 + 10t^2 - 3t$$

is an integer?

Problem 13 Let $F$ be a finite field with $q$ elements and let $V$ be an $n$-dimensional vector space over $F$.

1. Determine the number of elements in $V$.
2. Let $GL_n(F)$ denote the group of all $n \times n$ nonsingular matrices over $F$. Determine the order of $GL_n(F)$.
3. Let $SL_n(F)$ denote the subgroup of $GL_n(F)$ consisting of matrices with determinant 1. Find the order of $SL_n(F)$. 
Problem 14 Let $A$, $B$, and $C$ be finite abelian groups such that $A \times B$ and $A \times C$ are isomorphic. Prove that $B$ and $C$ are isomorphic.

Problem 15 Show that
\[ \sum_{n=0}^{\infty} \frac{z}{(1 + z^2)^n} \]
converges for all complex numbers $z$ exterior to the lemniscate $|1 + z^2| = 1$.

Problem 16 Let $g_n(z)$ be an entire function having only real zeros, $n = 1, 2, \ldots$. Suppose
\[ \lim_{n \to \infty} g_n(z) = g(z) \]
uniformly on compact sets in $\mathbb{C}$, with $g$ not identically zero. Prove that $g(z)$ has only real zeros.

Problem 17 Let $f : \mathbb{R}^3 \to \mathbb{R}$ be such that
\[ f^{-1}(0) = \{ v \in \mathbb{R}^3 \mid \|v\| = 1 \} \].
Suppose $f$ has continuous partial derivatives of orders $\leq 2$. Is there a $p \in \mathbb{R}^3$ with $\|p\| \leq 1$ such that
\[ \frac{\partial^2 f}{\partial x^2}(p) + \frac{\partial^2 f}{\partial y^2}(p) + \frac{\partial^2 f}{\partial z^2}(p) \geq 0 \]?

Problem 18 Let $E$ be a three-dimensional vector space over $\mathbb{Q}$. Suppose $T : E \to E$ is a linear transformation and $Tx = y$, $Ty = z$, $Tz = x + y$, for certain $x, y, z \in E$, $x \neq 0$. Prove that $x$, $y$, and $z$ are linearly independent.

Problem 19 Let $\{f_n\}$ be a sequence of continuous real functions defined on $[0, 1]$ such that
\[ \int_0^1 (f_n(y))^2 \, dy \leq 5 \]
for all $n$. Define $g_n : [0, 1] \to \mathbb{R}$ by
\[ g_n(x) = \int_0^1 \sqrt{x + yf_n(y)} \, dy. \]
1. Find a constant $K \geq 0$ such that $|g_n(x)| \leq K$ for all $n$.

2. Prove that a subsequence of the sequence $\{g_n\}$ converges uniformly.

**Problem 20** Let $x : \mathbb{R} \to \mathbb{R}$ be a solution to the differential equation

$$5x'' + 10x' + 6x = 0.$$

Prove that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = \frac{x(t)^2}{1 + x(t)^4}$$

attains a maximum value.