Preliminary Exam - Summer 1978

Problem 1 *For each of the following either give an example or else prove that no such example is possible.*

1. *A nonabelian group.*
2. *A finite abelian group that is not cyclic.*
3. *An infinite group with a subgroup of index* 5.
4. *Two finite groups that have the same order but are not isomorphic.*
5. *A group G with a subgroup H that is* not *normal.*
6. *A nonabelian group with no normal subgroups except the whole group and the unit element.*
7. *A group G with a normal subgroup H such that the factor group G/H is* not *isomorphic to any subgroup of G.*
8. *A group G with a subgroup H which has index 2 but is* not *normal.*

Problem 2 *Let R be the set of 2×2 matrices of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a, b are elements of a given field $\mathbf{F}$. Show that with the usual matrix operations, R is a commutative ring with identity. For which of the following fields $\mathbf{F}$ is R a field: $\mathbf{F} = \mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_5$, $\mathbb{Z}_7$?

Problem 3 *Let A be a $n \times n$ real matrix.*

1. *If the sum of each column element of A is 1 prove that there is a nonzero column vector x such that $Ax = x$.*
2. *Suppose that $n = 2$ and all entries in A are positive. Prove there is a nonzero column vector y and a number $\lambda > 0$ such that $Ay = \lambda y$.*

Problem 4 1. *Using only the axioms for a field* $\mathbf{F}$*, prove that a system of* m *homogeneous linear equations in* n *unknowns with* $m < n$ *and coefficients in* F *has a nonzero solution.*

2. *Use (1) to show that if* V *is a vector space over* $\mathbf{F}$ *which is spanned by a finite number of elements, then every maximal linearly independent subset of* V *has the same number of elements.*

Problem 5 *Evaluate*

$$\int_0^{2\pi} e^{(e^{i\theta}-i\theta)}\,d\theta\,.$$

Problem 6 *Let* $f : \mathbb{C} \to \mathbb{C}$ *be an entire function and let* $a > 0$ *and* $b > 0$ *be constants.*

1. *If* $|f(z)| \leqslant a\sqrt{|z|} + b$ *for all* z*, prove that* f *is a constant.*

2. *What can one prove about* f *if*

$$|f(z)| \leqslant a|z|^{5/2} + b$$

for all z*?*

Problem 7 1. *Solve the differential equation* $g' = 2g$*,* $g(0) = a$ *where* a *is a real constant.*

2. *Suppose* $f : [0,1] \to \mathbb{R}$ *is continuous with* $f(0) = 0$*, and for* $0 < x < 1$ f *is differentiable and* $0 \leqslant f'(x) \leqslant 2f(x)$*. Prove that* f *is identically* 0*.*

Problem 8 *Let* $\{S_\alpha\}$ *be a family of connected subsets of* $\mathbb{R}^2$ *all containing the origin. Prove that* $\bigcup_\alpha S_\alpha$ *is connected.*

Problem 9 *Let* X *and* Y *be nonempty subsets of a metric space* M*. Define*

$$d(X,Y) = \inf\{d(x,y) \mid x \in X, y \in Y\}.$$

1. *Suppose* X *contains only one point* x*, and* Y *is closed. Prove*

$$d(X,Y) = d(x,y)$$

for some $y \in Y$*.*

2. *Suppose X is compact and Y is closed. Prove*

$$d(X,Y) = d(x,y)$$

for some $x \in X$, $y \in Y$.

3. *Show by example that the conclusion of Part 2 can be false if X and Y are closed but not compact.*

Problem 10 *Let $U \subset \mathbb{R}^n$ be a convex open set and $f : U \to \mathbb{R}^n$ a differentiable function whose partial derivatives are uniformly bounded but not necessarily continuous. Prove that f has a unique continuous extension to the closure of U.*

Problem 11 *Suppose the power series*

$$\sum_{n=0}^{\infty} a_n z^n$$

converges for $|z| < R$ where z and the a_n are complex numbers. If $b_n \in \mathbb{C}$ are such that $|b_n| < n^2 |a_n|$ for all n, prove that

$$\sum_{n=0}^{\infty} b_n z^n$$

converges for $|z| < R$.

Problem 12

1. *Suppose f is analytic on a connected open set$U \subset \mathbb{C}$ and f takes only real values. Prove that f is constant.*

2. *Suppose $W \subset \mathbb{C}$ is open, g is analytic on W, and $g'(z) \neq 0$ for all $z \in W$. Show that*

$$\{\Re g(z) + \Im g(z) \mid z \in W\} \subset \mathbb{R}$$

is an open subset of $\mathbb{R}$.

Problem 13 *Let R denote the ring of polynomials over a field $\mathbf{F}$. Let $p_1, \ldots, p_n$ be elements of R. Prove that the greatest common divisor of $p_1, \ldots, p_n$ is 1 if and only if there is an $n \times n$ matrix over R of determinant 1 whose first row is $(p_1, \ldots, p_n)$.*

Problem 14 *Let G be a finite multiplicative group of 2×2 integer matrices.*

1. *Let $A \in G$. What can one prove about*

   (i) $\det A$*?*

   (ii) the (real or complex) eigenvalues of A?

   (iii) the Jordan or Rational Canonical Form of A?

   (iv) the order of A?

2. *Find all such groups up to isomorphism.*

Note: See also Problem **??**.

Problem 15 *Let V be a finite-dimensional vector space over an algebraically closed field. A linear operator $T : V \to V$ is called completely reducible if whenever a linear subspace $E \subset V$ is invariant under T, that is $T(E) \subset E$, there is a linear subspace $F \subset V$ which is invariant under T and such that $V = E \oplus F$. Prove that T is completely reducible if and only if V has a basis of eigenvectors.*

Problem 16

1. *Prove that a linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$ is diagonalizable if for all $\lambda \in \mathbb{C}$, $\ker(T - \lambda I)^n = \ker(T - \lambda I)$, where I is the $n \times n$ identity matrix.*

2. *Show that T is diagonalizable if T commutes with its conjugate transpose T^* (i.e., $(T^*)_{jk} = \overline{T_{kj}}$).*

Problem 17 *Let E be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are solutions to the differential equation $f''' + f'' - 2f = 0$.*

1. *Prove that E is a vector space and find its dimension.*

2. *Let $E_0 \subset E$ be the subspace of solutions g such that $\lim_{t\to\infty} g(t) = 0$. Find $g \in E_0$ such that $g(0) = 0$ and $g'(0) = 2$.*

Problem 18 *Let N be a* norm *on the vector space $\mathbb{R}^n$; that is, $N : \mathbb{R}^n \to \mathbb{R}$ satisfies*

$$\begin{aligned} &N(x) \geqslant 0 \text{ and } N(x) = 0 \text{ only if } x = 0,\\ &N(x+y) \leqslant N(x) + N(y),\\ &N(\lambda x) = |\lambda| N(x) \end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

1. *Prove that N is bounded on the unit sphere.*

2. *Prove that N is continuous.*

3. *Prove that there exist constants $A > 0$ and $B > 0$, such that for all $x \in \mathbb{R}^n, A\|x\| \leqslant N(x) \leqslant B\|x\|$.*

Problem 19 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $\mathbb{R}$ contains a countably infinite subset S such that*

$$\int_p^q f(x)\, dx = 0$$

if p and q are not in S. Prove that f is identically 0.

Problem 20 *Let $M_{n\times n}$ denote the vector space of real $n\times n$ matrices. Define a map $f{:}M_{n\times n} \to M_{n\times n}$ by $f(X) = X^2$. Find the derivative of f.*