Problem 1 For each of the following either give an example or else prove that no such example is possible.

1. A nonabelian group.
2. A finite abelian group that is not cyclic.
3. An infinite group with a subgroup of index 5.
4. Two finite groups that have the same order but are not isomorphic.
5. A group $G$ with a subgroup $H$ that is not normal.
6. A nonabelian group with no normal subgroups except the whole group and the unit element.
7. A group $G$ with a normal subgroup $H$ such that the factor group $G/H$ is not isomorphic to any subgroup of $G$.
8. A group $G$ with a subgroup $H$ which has index 2 but is not normal.

Problem 2 Let $R$ be the set of $2 \times 2$ matrices of the form

\[
\begin{pmatrix}
a & -b \\
b & a
\end{pmatrix}
\]

where $a$, $b$ are elements of a given field $F$. Show that with the usual matrix operations, $R$ is a commutative ring with identity. For which of the following fields $F$ is $R$ a field: $F = \mathbb{Q}$, $\mathbb{C}$, $\mathbb{Z}_5$, $\mathbb{Z}_7$?

Problem 3 Let $A$ be a $n \times n$ real matrix.

1. If the sum of each column element of $A$ is 1 prove that there is a nonzero column vector $x$ such that $Ax = x$.

2. Suppose that $n = 2$ and all entries in $A$ are positive. Prove there is a nonzero column vector $y$ and a number $\lambda > 0$ such that $Ay = \lambda y$. 
Problem 4  
1. Using only the axioms for a field $F$, prove that a system of $m$ homogeneous linear equations in $n$ unknowns with $m < n$ and coefficients in $F$ has a nonzero solution.

2. Use (1) to show that if $V$ is a vector space over $F$ which is spanned by a finite number of elements, then every maximal linearly independent subset of $V$ has the same number of elements.

Problem 5  
Evaluate 
\[ \int_0^{2\pi} e^{(e^{i\theta} - i\theta)} d\theta. \]

Problem 6  
Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function and let $a > 0$ and $b > 0$ be constants.

1. If $|f(z)| \leq a\sqrt{|z|} + b$ for all $z$, prove that $f$ is a constant.

2. What can one prove about $f$ if

\[ |f(z)| \leq a|z|^{5/2} + b \]

for all $z$?

Problem 7  
1. Solve the differential equation $g' = 2g$, $g(0) = a$ where $a$ is a real constant.

2. Suppose $f : [0,1] \to \mathbb{R}$ is continuous with $f(0) = 0$, and for $0 < x < 1$ $f$ is differentiable and $0 \leq f'(x) \leq 2f(x)$. Prove that $f$ is identically 0.

Problem 8  
Let $\{S_\alpha\}$ be a family of connected subsets of $\mathbb{R}^2$ all containing the origin. Prove that $\bigcup_\alpha S_\alpha$ is connected.

Problem 9  
Let $X$ and $Y$ be nonempty subsets of a metric space $M$. Define 
\[ d(X, Y) = \inf\{d(x, y) \mid x \in X, y \in Y\}. \]

1. Suppose $X$ contains only one point $x$, and $Y$ is closed. Prove 
\[ d(X, Y) = d(x, y) \]

for some $y \in Y$. 
2

2. Suppose $X$ is compact and $Y$ is closed. Prove
\[ d(X, Y) = d(x, y) \]
for some $x \in X$, $y \in Y$.

3. Show by example that the conclusion of Part 2 can be false if $X$ and $Y$ are closed but not compact.

**Problem 10** Let $U \subset \mathbb{R}^n$ be a convex open set and $f : U \to \mathbb{R}^n$ a differentiable function whose partial derivatives are uniformly bounded but not necessarily continuous. Prove that $f$ has a unique continuous extension to the closure of $U$.

**Problem 11** Suppose the power series
\[ \sum_{n=0}^{\infty} a_n z^n \]
converges for $|z| < R$ where $z$ and the $a_n$ are complex numbers. If $b_n \in \mathbb{C}$ are such that $|b_n| < n^2|a_n|$ for all $n$, prove that
\[ \sum_{n=0}^{\infty} b_n z^n \]
converges for $|z| < R$.

**Problem 12**
1. Suppose $f$ is analytic on a connected open set $U \subset \mathbb{C}$ and $f$ takes only real values. Prove that $f$ is constant.

2. Suppose $W \subset \mathbb{C}$ is open, $g$ is analytic on $W$, and $g'(z) \neq 0$ for all $z \in W$. Show that
\[ \{ \mathfrak{R}g(z) + \mathfrak{I}g(z) \mid z \in W \} \subset \mathbb{R} \]
is an open subset of $\mathbb{R}$.

**Problem 13** Let $R$ denote the ring of polynomials over a field $\mathbf{F}$. Let $p_1, \ldots, p_n$ be elements of $R$. Prove that the greatest common divisor of $p_1, \ldots, p_n$ is 1 if and only if there is an $n \times n$ matrix over $R$ of determinant 1 whose first row is $(p_1, \ldots, p_n)$.
Problem 14  Let $G$ be a finite multiplicative group of $2 \times 2$ integer matrices.

1. Let $A \in G$. What can one prove about
   
   (i) $\det A$?
   (ii) the (real or complex) eigenvalues of $A$?
   (iii) the Jordan or Rational Canonical Form of $A$?
   (iv) the order of $A$?

2. Find all such groups up to isomorphism.

Note: See also Problem ??.

Problem 15  Let $V$ be a finite-dimensional vector space over an algebraically closed field. A linear operator $T : V \to V$ is called completely reducible if whenever a linear subspace $E \subset V$ is invariant under $T$, that is $T(E) \subset E$, there is a linear subspace $F \subset V$ which is invariant under $T$ and such that $V = E \oplus F$. Prove that $T$ is completely reducible if and only if $V$ has a basis of eigenvectors.

Problem 16  

1. Prove that a linear operator $T : \mathbb{C}^n \to \mathbb{C}^n$ is diagonalizable if for all $\lambda \in \mathbb{C}$, $\ker(T - \lambda I)^n = \ker(T - \lambda I)$, where $I$ is the $n \times n$ identity matrix.

2. Show that $T$ is diagonalizable if $T$ commutes with its conjugate transpose $T^*$ (i.e., $(T^*)_{jk} = \overline{T_{kj}}$).

Problem 17  Let $E$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which are solutions to the differential equation $f''' + f'' - 2f = 0$.

1. Prove that $E$ is a vector space and find its dimension.

2. Let $E_0 \subset E$ be the subspace of solutions $g$ such that $\lim_{t \to \infty} g(t) = 0$. Find $g \in E_0$ such that $g(0) = 0$ and $g'(0) = 2$.

Problem 18  Let $N$ be a norm on the vector space $\mathbb{R}^n$; that is, $N : \mathbb{R}^n \to \mathbb{R}$ satisfies

\[
N(x) \geq 0 \text{ and } N(x) = 0 \text{ only if } x = 0, \\
N(x + y) \leq N(x) + N(y), \\
N(\lambda x) = |\lambda|N(x)
\]

for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.
1. Prove that $N$ is bounded on the unit sphere.

2. Prove that $N$ is continuous.

3. Prove that there exist constants $A > 0$ and $B > 0$, such that for all $x \in \mathbb{R}^n$, $A\|x\| \leq N(x) \leq B\|x\|$.

**Problem 19** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $\mathbb{R}$ contains a countably infinite subset $S$ such that

$$\int_p^q f(x) \, dx = 0$$

if $p$ and $q$ are not in $S$. Prove that $f$ is identically 0.

**Problem 20** Let $M_{n\times n}$ denote the vector space of real $n \times n$ matrices. Define a map $f : M_{n\times n} \to M_{n\times n}$ by $f(X) = X^2$. Find the derivative of $f$. 