Preliminary Exam - Summer 1977

Problem 1 *Prove the following statements about the polynomial ring* $\mathbf{F}[x]$*, where* $\mathbf{F}$ *is any field.*

1. $\mathbf{F}[x]$ *is a vector space over* $\mathbf{F}$.
2. *The subset* $\mathbf{F}_n[x]$ *of polynomials of degree* $\leqslant n$ *is a subspace of dimension* $n+1$ *in* $\mathbf{F}[x]$.
3. *The polynomials* $1, x-a, \ldots, (x-a)^n$ *form a basis of* $\mathbf{F}_n[x]$ *for any* $a \in \mathbf{F}$.

Problem 2 *Let* f *be continuous on* $\mathbb{C}$ *and analytic on* $\{z \mid \Im z \neq 0\}$. *Prove that* f *must be analytic on* $\mathbb{C}$.

Problem 3 *Prove that* $\alpha = \sqrt{3}+\sqrt{5}$ *is algebraic over* $\mathbb{Q}$, *by explicitly finding a polynomial with coefficients in* $\mathbb{Q}$ *of which* α *is a root.*

Problem 4 *Let* A *be an* $r \times r$ *matrix of real numbers. Prove that the infinite sum*

$$e^A = I + A + \frac{A^2}{2} + \cdots + \frac{A^n}{n!} + \cdots$$

of matrices converges (i.e., for each i, j*, the sum of* $(i,j)^{th}$ *entries converges), and hence that* e^A *is a well-defined matrix.*

Problem 5 *Write all values of* i^i *in the form* $a + bi$.

Problem 6 *Show that*

$$F(k) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\cos^2 x}}$$

$0 \leqslant k < 1$, *is an increasing function of* k.

Problem 7 *Let* $A : \mathbb{R}^6 \to \mathbb{R}^6$ *be a linear transformation such that* $A^{26} = I$. *Show that* $\mathbb{R}^6 = V_1 \oplus V_2 \oplus V_3$, *where* V_1, V_2, *and* V_3 *are two-dimensional invariant subspaces for* A.

Problem 8 *Prove that the initial value problem*

$$\frac{dx}{dt} = 3x + 85\cos x, \qquad x(0) = 77\,,$$

has a solution $x(t)$ defined for all $t \in \mathbb{R}$.

Problem 9 *Show that every rotation of $\mathbb{R}^3$ has an* axis*; that is, given a 3×3 real matrix A such that $A^t = A^{-1}$ and $\det A > 0$, prove that there is a nonzero vector v such that $Av = v$.*

Problem 10 *Suppose that $f(x)$ is defined on $[-1, 1]$, and that $f'''(x)$ is continuous. Show that the series*

$$\sum_{n=1}^{\infty} \left(n \left(f\left(\frac{1}{n}\right) - f\left(-\frac{1}{n}\right)\right) - 2f'(0)\right)$$

converges.

Problem 11 *Let $f(x, t)$ be a C^1 function such that $\partial f/\partial x = \partial f/\partial t$. Suppose that $f(x, 0) > 0$ for all x. Prove that $f(x, t) > 0$ for all x and t.*

Problem 12 *Let V be the vector space of all polynomials of degree $\leqslant 10$, and let D be the differentiation operator on V (i.e., $Dp(x) = p'(x)$).*

1. *Show that* $\operatorname{tr} D = 0$.
2. *Find all eigenvectors of D and e^D.*

Problem 13 *Let f be an analytic function such that*

$$f(z) = 1 + 2z + 3z^2 + \cdots \quad \text{for } |z| < 1.$$

Define a sequence of real numbers $a_0, a_1, a_2, \ldots$ by

$$f(z) = \sum_{n=0}^{\infty} a_n (z+2)^n.$$

What is the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n z^n?$$

Problem 14 *1. Prove that every finitely generated subgroup of $\mathbb{Q}$, the additive group of rational numbers, is cyclic.*

2. *Does the same conclusion hold for finitely generated subgroups of $\mathbb{Q}/\mathbb{Z}$, where $\mathbb{Z}$ is the group of integers?*

Note: See also Problems **??** *and* **??**.

Problem 15 *Let $A \subset \mathbb{R}^n$ be compact, $x \in A$; let (x_i) be a sequence in A such that every convergent subsequence of (x_i) converges to x.*

1. *Prove that the entire sequence (x_i) converges.*
2. *Give an example to show that if A is not compact, the result in Part 1 is not necessarily true.*

Problem 16 *Use the Residue Theorem to evaluate the integral*

$$I(a) = \int_0^{2\pi} \frac{d\theta}{a + \cos\theta}$$

where a is real and $a > 1$. Why the formula obtained for $I(a)$ is also valid for certain complex (nonreal) values of a ?

Problem 17 *In the ring $\mathbb{Z}[x]$ of polynomials in one variable over the integers, show that the ideal $\mathfrak{I}$ generated by 5 and $x^2 + 2$ is a maximal ideal.*

Problem 18 *Let $\hat{a}_0 + \hat{a}_1 z + \cdots + \hat{a}_n z^n$ be a polynomial having $\hat{z}$ as a simple root. Show that there is a continuous function $r : U \to \mathbb{C}$, where U is a neighborhood of $(\hat{a}_0, \ldots, \hat{a}_n)$ in $\mathbb{C}^{n+1}$, such that $r(a_0, \ldots, a_n)$ is always a root of $a_0 + a_1 z + \cdots + a_n z^n$, and $r(\hat{a}_0, \ldots, \hat{a}_n) = \hat{z}$.*

Problem 19 *Let p be an odd prime. If the congruence $x^2 \equiv -1 \pmod{p}$ has a solution, show that $p \equiv 1 \pmod{4}$.*

Problem 20 *Determine all solutions to the following infinite system of linear equations in the infinitely many unknowns $x_1, x_2, \ldots$:*

$$\begin{array}{ccccccc} x_1 & + & x_3 & + & x_5 & = & 0 \\ x_2 & + & x_4 & + & x_6 & = & 0 \\ x_3 & + & x_5 & + & x_7 & = & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

How many free parameters are required?