Preliminary Exam - Spring 1999

Problem 1 *Let $d(X,Y)$ the distance between X and Y as defined in Problem* **??**. *Give a proof or counterexample for each of the following statements, for disjoint sets X and Y.*

1. *If X and Y are closed in M then $d(X,Y) > 0$.*
2. *If X and Y are compact then $d(X,Y) > 0$.*
3. *If X is closed and Y compact then $d(X,Y) > 0$.*

Problem 2 *Suppose that a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly on $\mathbb{R}$ to a function $f : \mathbb{R} \to \mathbb{R}$, and that $c_n = \lim\limits_{x\to\infty} f_n(x)$ exists for each positive integer n. Prove that $\lim\limits_{n\to\infty} c_n$ and $\lim\limits_{x\to\infty} f(x)$ both exist and are equal.*

Problem 3 *Suppose that f is a twice differentiable real-valued function on $\mathbb{R}$ such that $f(0) = 0$, $f'(0) > 0$, and $f''(x) \geqslant f(x)$ for all $x \geqslant 0$. Prove that $f(x) > 0$ for all $x > 0$.*

Problem 4 *Evaluate* $\displaystyle\int_0^\infty \frac{dx}{x^c(x+1)}$ *for each real number $c \in (0,1)$.*

Problem 5

1. *Prove that if f is holomorphic on the unit disc $\mathbb{D}$ and $f(z) \neq 0$ for all $z \in \mathbb{D}$, then there is a holomorphic function g on $\mathbb{D}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{D}$.*
2. *Does the conclusion of Part 1 remain true if $\mathbb{D}$ is replaced by an arbitrary connected open set in $\mathbb{C}$?*

Problem 6 *Let p, q, r and s be polynomials of degree at most 3. Which, if any, of the following two conditions is sufficient for the conclusion that the polynomials are linearly dependent?*

1. *At 1 each of the polynomials has the value 0.*
2. *At 0 each of the polynomials has the value 1.*

Problem 7 *Suppose that the minimal polynomial of a linear operator T on a seven-dimensional vector space is x^2. What are the possible values of the dimension of the kernel of T?*

Problem 8 *Let M be a 3×3 matrix with entries in the polynomial ring $\mathbb{R}[t]$ such that $M^3 = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$. Let N be the matrix with real entries obtained by substituting $t = 0$ in M. Prove that N is similar to $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.*

Problem 9 *Let G be a finite group, with identity e. Suppose that for every $a, b \in G$ distinct from e, there is an automorphism σ of G such that$\sigma(a) = b$. Prove that G is abelian.*

Problem 10 *Suppose that f is a twice differentiable real function such that $f''(x) > 0$ for all $x \in [a, b]$. Find all numbers $c \in [a, b]$ at which the area between the graph $y = f(x)$, the tangent to the graph at $(c, f(c))$, and the lines $x = a$, $x = b$, attains its minimum value.*

Problem 11 *Prove that if n is a positive integer and α, ε are real numbers with $\varepsilon > 0$, then there is a real function f with derivatives of all orders such that*

1. $|f^{(k)}(x)| \leqslant \varepsilon$ *for* $k = 0, 1, \ldots, n-1$ *and all* $x \in \mathbb{R}$,
2. $f^{(k)}(0) = 0$ *for* $k = 0, 1, \ldots, n-1$,
3. $f^{(n)}(0) = \alpha$.

Problem 12 *Suppose that f is holomorphic on some neighborhood of a in the complex plane. Prove that either f is constant on some neighborhood of a, or there exist an integer $n > 0$ and real numbers $\delta, \varepsilon > 0$ such that for each complex number b satisfying $0 < |b - f(a)| < \varepsilon$, the equation $f(z) = b$ has exactly n roots in $\{z \in \mathbb{C} \mid |z - a| < \delta\}$.*

Problem 13 *Let b_1, b_2, $\ldots$ be a sequence of real numbers such that $b_k \geqslant b_{k+1}$ for all k and $\lim_{k\to\infty} b_k = 0$. Prove that the power series $\sum_{k=1}^{\infty} b_k z^k$ converges for all complex numbers z such that $|z| \leqslant 1$ and $z \neq 1$.*

Problem 14 *Let* $A = (a_{ij})$ *be a* $n \times n$ *complex matrix such that*$a_{ij} \neq 0$ *if* $i = j + 1$ *but* $a_{ij} = 0$ *if* $i \geqslant j + 2$. *Prove that* A *cannot have more than one Jordan block for any eigenvalue.*

Problem 15 *Let* M *be a square complex matrix, and let*$S = \{XMX^{-1} |\ X \text{is non-singular}\}$ *be the set of all matrices similar to* M. *Show that* M *is a nonzero multiple of the identity matrix if and only if no matrix in* S *has a zero anywhere on its diagonal.*

Problem 16 *Let* $\|x\|$ *denote the Euclidean norm of a vector* x. *Show that for any real* $m \times n$ *matrix* M *there is a unique non-negative scalar* σ, *and (possibly non-unique) unit vectors* $u \in \mathbb{R}^n$ *and* $v \in \mathbb{R}^m$ *such that*

1. $\|Mx\| \leqslant \sigma \|x\|$ *for all* $x \in \mathbb{R}^n$,
2. $Mu = \sigma v$,
3. $M^T v = \sigma u$ *(where* M^T *is the transpose of* M*).*

Problem 17 *Let* $f(x) \in \mathbb{Q}[x]$ *be an irreducible polynomial of degree* $n \geqslant 3$. *Let* L *be the splitting field of* f, *and let* $\alpha \in L$ *be a zero of* f. *Given that* $[L : \mathbb{Q}] = n!$, *prove that* $\mathbb{Q}(\alpha^4) = \mathbb{Q}(\alpha)$.

Problem 18 *Let* G *be a finite simple group of order* n. *Determine the number of normal subgroups in the direct product* $G \times G$.