Preliminary Exam - Spring 1998

Problem 1 *Prove that the polynomial* $z^4 + z^3 + 1$ *has exactly one root in the quadrant* $\{z = x + iy \mid x, y > 0\}$.

Problem 2 *Let* f *be analytic in an open set containing the closed unit disc. Suppose that* $|f(z)| > m$ *for* $|z| = 1$ *and* $|f(0)| < m$. *Prove that* $f(z)$ *has at least one zero in the open unit disc* $|z| < 1$.

Problem 3 *Let* M *be a non-empty complete metric space. Let* $T : M \to M$ *such that* $T \circ T = T^2$ *is a strict contraction. Prove that* T *has a unique fixed point in* M, *i.e., there is a unique point* x_0 *with* $T(x_0) = x_0$.

Problem 4 *Using the properties of the Riemann integral, show that if* f *is a non-negative continuous function on* $[0, 1]$, *and* $\int_0^1 f(x)dx = 0$, *then* $f(x) = 0$ *for all* $x \in [0, 1]$.

Problem 5 *Let* A *be the ring of real 2×2 matrices of the form* $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$. *What are the 2-sided ideals in* A?

Problem 6 *Let* G *be the group* $\mathbb{Q}/\mathbb{Z}$. *Show that for every positive integer* t, G *has a unique cyclic subgroup of order* t.

Problem 7 *Suppose that* A *and* B *are two commuting* $n \times n$ *complex matrices. Show that they have a common eigenvector.*

Problem 8 *Let* $m \geqslant 0$ *be an integer. Let* $a_1, a_2, \ldots, a_m$ *be integers and let*

$$f(x) = \sum_{i=1}^{m} \frac{a_i x^i}{i!}$$

Show that if $d \geqslant 0$ *is an integer then* $f(x)^d/d!$ *can be expressed in the form*

$$\sum_{i=0}^{md} \frac{b_i x^i}{i!}\,.$$

where the b_i *are integers.*

Problem 9 *Let* $M_1 = \left(\begin{smallmatrix} 3 & 2 \\ 1 & 4 \end{smallmatrix}\right)$, $M_2 = \left(\begin{smallmatrix} 5 & 7 \\ -3 & -4 \end{smallmatrix}\right)$, $M_3 = \left(\begin{smallmatrix} 5 & 6.9 \\ -3 & -4 \end{smallmatrix}\right)$. *For which (if any)* i, $1 \leqslant i \leqslant 3$, *is the sequence* (M_i^n) *bounded away from* ∞? *For which* i *is the sequence bounded away from* 0?

Problem 10 *Let* $P_i = (a\cos\theta_i, b\sin\theta_i)$, $i = 1, 2, 3$, *be a triple of points on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *What is the maximal value of the area of the triangle* $\Delta P_1 P_2 P_3$? *Determine those triangles with maximal area.*

Problem 11 *Let* $A, B, \ldots, F$ *be real coefficients. Show that thequadratic form*

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2$$

is positive definite if and only if

$$A > 0, \qquad \begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0, \qquad \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} > 0.$$

Problem 12 *Given the fact that* $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, *evaluate the integral*

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2 + (y-x)^2 + y^2)} dx \; dy \; .$$

Problem 13 *Let* a *be a complex number with* $|a| < 1$. *Evaluate the integral*

$$\int_{|z|=1} \frac{|dz|}{|z-a|^2}$$

Problem 14 *Let* K *be a real constant. Suppose that* $y(t)$ *is a positive differentiable function satisfying* $y'(t) \leqslant Ky(t)$ *for* $t \geqslant 0$. *Prove that*$y(t) \leqslant e^{Kt} y(0)$ *for* $t \geqslant 0$.

Problem 15 *For continuous real valued functions* f, g *on the interval* $[-1, 1]$ *define the inner product* $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$. *Find that polynomial of the form* $p(x) = a + bx^2 - x^4$ *which is orthogonal on* $[-1, 1]$ *to all lower order polynomials.*

Problem 16 *Let $a > 0$. Show that the complex function*

$$f(z) = \frac{1 + z + az^2}{1 - z + az^2}$$

satisfies $|f(z)| < 1$ for all z in the open left half-plane $\Re z < 0$.

Problem 17 *Let A be an $n \times n$ complex matrix with* $\operatorname{tr} A = 0$. *Show that A is similar to a matrix with all 0's along the main diagonal.*

Problem 18 *Let N be a nilpotent complex matrix. Let r be a positive integer. Show that there is a $n \times n$ complex matrix A with*

$$A^r = I + N.$$