Problem 1  For which values of the exponents $\alpha, \beta$ does the following series converge?

$$\sum_{n=3}^{\infty} \frac{1}{n^\alpha (\log n)^\beta}.$$ 

Problem 2  Let $M$ be a metric space with metric $d$. Let $C$ be a nonempty closed subset of $M$. Define $f : M \to \mathbb{R}$ by

$$f(x) = \inf \{d(x, y) \mid y \in C\}.$$ 

Show that $f$ is continuous, and that $f(x) = 0$ if and only if $x \in C$.

Problem 3  Suppose that $f(x)$ is continuous, nonnegative for $x \geq 0$, with $\int_{0}^{\infty} f(x) dx < \infty$. Prove that

$$\lim_{n \to \infty} \int_{0}^{n} \frac{x f(x)}{n} dx = 0.$$ 

Problem 4  Let $f$ and $g$ be two entire functions such that, for all $z \in \mathbb{C}$, $\Re f(z) \leq k \Re g(z)$ for some real constant $k$ (independent of $z$). Show that there are constants $a, b$ such that

$$f(z) = ag(z) + b.$$ 

Problem 5  Prove that

$$\int_{-\infty}^{\infty} \frac{e^{-(t-i\gamma)^2/2}}{\sqrt{2\pi}} dt$$ 

is independent of the real parameter $\gamma$.

Problem 6  Suppose that $X$ is a topological space and $V$ is a finite-dimensional subspace of the vector space of continuous real valued functions on $X$. Prove that there exist a basis $\{f_1, \ldots, f_n\}$ for $V$ and points $x_1, \ldots, x_n$ in $X$ such that $f_i(x_j) = \delta_{ij}$.
Problem 7  Suppose that $A$ and $B$ are endomorphisms of a finite-dimensional vector space $V$ over a field $K$. Prove or disprove the following statements:

1. Every eigenvector of $AB$ is also an eigenvector of $BA$.
2. Every eigenvalue of $AB$ is also an eigenvalue of $BA$.

Problem 8  Classify all abelian groups of order 80 up to isomorphism.

Problem 9  Let $R$ be the ring of $n \times n$ matrices over a field. Suppose $S$ is a ring and $h : R \to S$ is a homomorphism. Show that $h$ is either injective or zero.

Problem 10  Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function (i.e., there is a constant $M$ such that $|f(x)| \leq M$ for all $x \in \mathbb{R}$). Suppose the graph of $f$ is a closed subset of $\mathbb{R}^2$. Prove that $f$ is continuous.

Problem 11  Suppose that $f''(x) = (x^2 - 1)f(x)$ for all $x \in \mathbb{R}$, and that $f(0) = 1$, $f'(0) = 0$. Show that $f(x) \to 0$ as $x \to \infty$.

Problem 12  Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx.$$ 

Problem 13  Suppose that $f : \mathbb{C} \to \mathbb{C}$ is injective and everywhere holomorphic. Prove that there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $f(z) = az + b$ for all $z \in \mathbb{C}$.

Problem 14  Show that

$$\det(\exp(M)) = e^{\text{tr}(M)}$$

for any complex $n \times n$ matrix $M$, where $\exp(M)$ is defined as in Problem ??.

Problem 15  Suppose that $P$ and $Q$ are $n \times n$ matrices such that $P^2 = P$, $Q^2 = Q$, and $1 - (P + Q)$ is invertible. Show that $P$ and $Q$ have the same rank.
Problem 16 Suppose that $A$ is a commutative algebra with identity over $\mathbb{C}$ (i.e., $A$ is a commutative ring containing $\mathbb{C}$ as a subring with identity). Suppose further that $a^2 \neq 0$ for all nonzero elements $a \in A$. Show that if the dimension of $A$ as a vector space over $\mathbb{C}$ is finite and at least two, then the equations $a^2 = a$ is satisfied by at least three distinct elements $a \in A$.

Problem 17 Let $GL_2(\mathbb{Z}_m)$ denote the multiplicative group of invertible $2 \times 2$ matrices over the ring of integers modulo $m$. Find the order of $GL_2(\mathbb{Z}_{p^n})$ for each prime $p$ and positive integer $n$.

Problem 18 Let $H$ be the quotient of an abelian group $G$ by a subgroup $K$. Prove or disprove each of the following statements:

1. If $H$ is finite cyclic then $G$ is isomorphic to the direct product of $H$ and $K$.

2. If $H$ is a direct product of infinite cyclic groups then $G$ is isomorphic to the direct product of $H$ and $K$. 