Preliminary Exam - Spring 1997

Problem 1 *For which values of the exponents α, β does the following series converge?*

$$\sum_{n=3}^{\infty} \frac{1}{n^{\alpha}(\log n)^{\beta}}\,.$$

Problem 2 *Let M be a metric space with metric d. Let C be a nonempty closed subset of M. Define $f : M \to \mathbb{R}$ by*

$$f(x) = \inf\{d(x,y) \,|\, y \in C\}.$$

Show that f is continuous, and that $f(x) = 0$ if and only if $x \in C$.

Problem 3 *Suppose that $f(x)$ is continuous, nonnegative for $x \geqslant 0$, with $\int_0^{\infty} f(x)dx < \infty$. Prove that*

$$\lim_{n\to\infty} \int_0^n \frac{xf(x)}{n}dx = 0\,.$$

Problem 4 *Let f and g be two entire functions such that, for all $z \in \mathbb{C}$, $\Re f(z) \leqslant k\Re g(z)$ for some real constant k (independent of z). Show that there are constants a, b such that*

$$f(z) = ag(z) + b\,.$$

Problem 5 *Prove that*

$$\int_{-\infty}^{\infty} \frac{e^{-(t-i\gamma)^2/2}}{\sqrt{2\pi}}dt$$

is independent of the real parameter γ.

Problem 6 *Suppose that X is a topological space and V is a finite-dimensional subspace of the vector space of continuous real valued functions on X. Prove that there exist a basis $\{f_1, \ldots, f_n\}$ for V and points $x_1, \ldots, x_n$ in X such that $f_i(x_j) = \delta_{ij}$.*

Problem 7 *Suppose that* A *and* B *are endomorphisms of a finite-dimensional vector space* V *over a field* $\mathbf{K}$*. Prove or disprove the following statements:*

1. *Every eigenvector of* AB *is also an eigenvector of* BA*.*
2. *Every eigenvalue of* AB *is also an eigenvalue of* BA*.*

Problem 8 *Classify all abelian groups of order* 80 *up to isomorphism.*

Problem 9 *Let* R *be the ring of* $n \times n$ *matrices over a field. Suppose* S *is a ring and* $h : R \to S$ *is a homomorphism. Show that* h *is either injective or zero.*

Problem 10 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a bounded function (i.e., there is a constant* M *such that* $|f(x)| \leqslant M$ *for all* $x \in \mathbb{R}$*). Suppose the graph of* f *is a closed subset of* $\mathbb{R}^2$*. Prove that* f *is continuous.*

Problem 11 *Suppose that* $f''(x) = (x^2 - 1)f(x)$ *for all* $x \in \mathbb{R}$*, and that* $f(0) = 1,\ f'(0) = 0$*. Show that* $f(x) \to 0$ *as* $x \to \infty$*.*

Problem 12 *Evaluate the integral*

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx\,.$$

Problem 13 *Suppose that* $f : \mathbb{C} \to \mathbb{C}$ *is injective and everywhere holomorphic. Prove that there exist* $a, b \in \mathbb{C}$ *with* $a \neq 0$ *such that* $f(z) = az + b$ *for all* $z \in \mathbb{C}$*.*

Problem 14 *Show that*

$$\det(\exp(M)) = e^{\mathrm{tr}(M)}$$

for any complex $n \times n$ *matrix* M*, where* $\exp(M)$ *is defined as in Problem* **??***.*

Problem 15 *Suppose that* P *and* Q *are* $n \times n$ *matrices such that*$P^2 = P$*,* $Q^2 = Q$*, and* $1 - (P + Q)$ *is invertible. Show that* P *and* Q *have the same rank.*

Problem 16 *Suppose that A is a commutative algebra with identity over $\mathbb{C}$ (i.e., A is a commutative ring containing $\mathbb{C}$ as a subring with identity). Suppose further that $a^2 \neq 0$ for all nonzero elements $a \in A$. Show that if the dimension of A as a vector space over $\mathbb{C}$ is finite and at least two, then the equations $a^2 = a$ is satisfied by at least three distinct elements $a \in A$.*

Problem 17 *Let $GL_2(\mathbb{Z}_m)$ denote the multiplicative group of invertible 2×2 matrices over the ring of integers modulo m. Find the order of $GL_2(\mathbb{Z}_{p^n})$ for each prime p and positive integer n.*

Problem 18 *Let H be the quotient of an abelian group G by a subgroup K. Prove or disprove each of the following statements:*

1. *If H is finite cyclic then G is isomorphic to the direct product of H and K.*

2. *If H is a direct product of infinite cyclic groups then G is isomorphic to the direct product of H and K.*