Problem 1 Compute
\[ L = \lim_{n \to \infty} \left( \frac{n^n}{n!} \right)^{1/n} . \]

Problem 2 Let \( K \subset \mathbb{R}^n \) be compact and \( \{B_j\}_{j=1}^{\infty} \) be a sequence of open balls which covers \( K \). Prove that there is \( \varepsilon > 0 \) such that every \( \varepsilon \)-ball centered at a point of \( K \) is contained in one of the balls \( B_j \).

Problem 3 Compute
\[ I = \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} . \]

Problem 4 Let \( r < 1 < R \). Show that for all sufficiently small \( \varepsilon > 0 \), the polynomial
\[ p(z) = \varepsilon z^7 + z^2 + 1 \]
has exactly five roots (counted with their multiplicities) inside the annulus
\[ r\varepsilon^{-1/5} < |z| < R\varepsilon^{-1/5} . \]

Problem 5 Prove or disprove: For any \( 2 \times 2 \) matrix \( A \) over \( \mathbb{C} \), there is a \( 2 \times 2 \) matrix \( B \) such that \( A = B^2 \).

Problem 6 If a finite homogeneous system of linear equations with rational coefficients has a nonzero complex solution, need it have a nonzero rational solution? Prove or give a counterexample.

Problem 7 Prove that \( f(x) = x^4 + x^3 + x^2 + 6x + 1 \) is irreducible over \( \mathbb{Q} \).

Problem 8 Determine the rightmost decimal digit of
\[ A = 17^{17^{17}} . \]

Problem 9 Exhibit infinitely many pairwise nonisomorphic quadratic extensions of \( \mathbb{Q} \) and show they are pairwise nonisomorphic.
Problem 10  Show that a positive constant $t$ can satisfy
\[ e^x > x^t \quad \text{for all} \quad x > 0 \]
if and only if $t < e$.

Problem 11  Suppose $\varphi$ is a $C^1$ function on $\mathbb{R}$ such that
\[ \varphi(x) \to a \quad \text{and} \quad \varphi'(x) \to b \quad \text{as} \quad x \to \infty. \]
Prove or give a counterexample: $b$ must be zero.

Problem 12  Let $M_{2 \times 2}$ be the space of $2 \times 2$ matrices over $\mathbb{R}$, identified in the usual way with $\mathbb{R}^4$. Let the function $F$ from $M_{2 \times 2}$ into $M_{2 \times 2}$ be defined by
\[ F(X) = X + X^2. \]
Prove that the range of $F$ contains a neighborhood of the origin.

Problem 13  Let $f = u + iv$ be analytic in a connected open set $D$, where $u$ and $v$ are real valued. Suppose there are real constants $a$, $b$ and $c$ such that
\[ a^2 + b^2 \neq 0 \]
and
\[ au + bv = c \]
in $D$. Show that $f$ is constant in $D$.

Problem 14  Suppose $f: [0, 1] \to \mathbb{C}$ is continuous. Show that
\[ g(z) = \int_0^1 f(t)e^{tz^2} dt \]
defines a function $g$ that is analytic everywhere in the complex plane.

Problem 15  Suppose that $A$ and $B$ are real matrices such that $A^t = A$,
\[ v^tAv \geq 0 \]
for all $v \in \mathbb{R}^n$ and
\[ AB + BA = 0. \]
Show that $AB = BA = 0$ and give an example where neither $A$ nor $B$ is zero.
Problem 16 Let $A$ be the $n \times n$ matrix which has zeros on the main diagonal and ones everywhere else. Find the eigenvalues and eigenspaces of $A$ and compute $\det(A)$.

Problem 17 Let $G$ be the group of $2 \times 2$ matrices with determinant 1 over the four-element field $F$. Let $S$ be the set of lines through the origin in $F^2$. Show that $G$ acts faithfully on $S$. (The action is faithful if the only element of $G$ which fixes every element of $S$ is the identity.)

Problem 18 Let $G$ and $H$ be finite groups of relatively prime orders. Show that the automorphism group $\text{Aut}(G \times H)$ is isomorphic to the direct product of $\text{Aut}(G)$ and $\text{Aut}(H)$. 