Preliminary Exam - Spring 1996

Problem 1 *Compute*

$$L = \lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n}.$$

Problem 2 *Let $K \subset \mathbb{R}^n$ be compact and $\{B_j\}_{j=1}^{\infty}$ be a sequence of open balls which covers K. Prove that there is $\varepsilon > 0$ such that every ε-ball centered at a point of K is contained in one of the balls B_j.*

Problem 3 *Compute*

$$I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}.$$

Problem 4 *Let $r < 1 < R$. Show that for all sufficiently small $\varepsilon > 0$, the polynomial*

$$p(z) = \varepsilon z^7 + z^2 + 1$$

has exactly five roots (counted with their multiplicities) inside the annulus

$$r\varepsilon^{-1/5} < |z| < R\varepsilon^{-1/5}.$$

Problem 5 *Prove or disprove: For any 2×2 matrix A over $\mathbb{C}$, there is a 2×2 matrix B such that $A = B^2$.*

Problem 6 *If a finite homogeneous system of linear equations with rational coefficients has a nonzero complex solution, need it have a nonzero rational solution? Prove or give a counterexample.*

Problem 7 *Prove that $f(x) = x^4 + x^3 + x^2 + 6x + 1$ is irreducible over $\mathbb{Q}$.*

Problem 8 *Determine the rightmost decimal digit of*

$$A = 17^{17^{17}}.$$

Problem 9 *Exhibit infinitely many pairwise nonisomorphicquadratic extensions of $\mathbb{Q}$ and show they are pairwise nonisomorphic.*

Problem 10 *Show that a positive constant* t *can satisfy*

$$e^x > x^t \quad for\ all \quad x > 0$$

if and only if $t < e$.

Problem 11 *Suppose* φ *is a* C^1 *function on* $\mathbb{R}$ *such that*

$$\varphi(x) \to a \quad and \quad \varphi'(x) \to b \quad as \quad x \to \infty.$$

Prove or give a counterexample: b *must be zero.*

Problem 12 *Let* $M_{2\times 2}$ *be the space of 2×2 matrices over* $\mathbb{R}$, *identified in the usual way with* $\mathbb{R}^4$. *Let the function* F *from* $M_{2\times 2}$ *into* $M_{2\times 2}$ *be defined by*

$$F(X) = X + X^2.$$

Prove that the range of F *contains a neighborhood of the origin.*

Problem 13 *Let* $f = u + iv$ *be analytic in a connected open set* D, *where* u *and* v *are real valued. Suppose there are real constants* a, b *and* c *such that* $a^2 + b^2 \neq 0$ *and*

$$au + bv = c$$

in D. *Show that* f *is constant in* D.

Problem 14 *Suppose* $f\colon [0,1] \to \mathbb{C}$ *is continuous. Show that*

$$g(z) = \int_0^1 f(t)e^{tz^2}dt$$

defines a function g *that is analytic everywhere in the complex plane.*

Problem 15 *Suppose that* A *and* B *are real matrices such that* $A^t = A$,

$$v^t A v \geqslant 0$$

for all $v \in \mathbb{R}^n$ *and*

$$AB + BA = 0.$$

Show that $AB = BA = 0$ *and give an example where neither* A *nor* B *is zero.*

Problem 16 *Let A be the $n \times n$ matrix which has zeros on the main diagonal and ones everywhere else. Find the eigenvalues and eigenspaces of A and compute* $\det(A)$.

Problem 17 *Let G be the group of 2×2 matrices with determinant* 1 *over the four-element field* $\mathbf{F}$. *Let S be the set of lines through the origin in* $\mathbf{F}^2$. *Show that G acts* faithfully *on S. (The action is faithful if the only element of G which fixes every element of S is the identity.)*

Problem 18 *Let G and H be finite groups of relatively prime orders. Show that the automorphism group* $\mathrm{Aut}(G \times H)$ *is isomorphic to the direct product of* $\mathrm{Aut}(G)$ *and* $\mathrm{Aut}(H)$.