Preliminary Exam - Spring 1995

Problem 1 *For each positive integer* n, *define* $f_n : \mathbb{R} \to \mathbb{R}$ *by* $f_n(x) = \cos nx$. *Prove that the sequence of functions* $\{f_n\}$ *has no uniformly convergent subsequence.*

Problem 2 *Let* A *be the 3×3 matrix*

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Determine all real numbers a *for which the limit* $\lim_{n\to\infty} a^n A^n$ *exists and is nonzero (as a matrix).*

Problem 3 *Let* n *be a positive integer and* $0 < \theta < \pi$. *Prove that*

$$\frac{1}{2\pi i}\int_{|z|=2} \frac{z^n}{1 - 2z\cos\theta + z^2}\,dz = \frac{\sin n\theta}{\sin\theta}$$

where the circle $|z| = 2$ *is oriented counterclockwise.*

Problem 4 *Let* $\mathbf{F}$ *be a finite field of cardinality* p^n, *with* p *prime and* $n > 0$, *and let* G *be the group of invertible 2×2 matrices with coefficients in* $\mathbf{F}$.

1. *Prove that* G *has order* $(p^{2n} - 1)(p^{2n} - p^n)$.
2. *Show that any* p*-Sylow subgroup of* G *is isomorphic to the additive group of* $\mathbf{F}$.

Problem 5 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a bounded continuously differentiable function. Show that every solution of* $y'(x) = f(y(x))$ *is monotone.*

Problem 6 *Suppose that* R *is a subring of a commutative ring* S *and that* R *is of finite index* n *in* S. *Let* m *be an integer that is relatively prime to* n. *Prove that the natural map* $R/mR \to S/mS$ *is a ring isomorphism.*

Problem 7 *Let $f,\ g\colon[0,1]\to[0,\infty)$ be continuous functions satisfying*

$$\sup_{0\leqslant x\leqslant 1} f(x) = \sup_{0\leqslant x\leqslant 1} g(x).$$

Prove that there exists $t\in[0,1]$ with $f(t)^2+3f(t)=g(t)^2+3g(t)$.

Problem 8 *Suppose that $W\subset V$ are finite-dimensional vector spaces over a field, and let $L\colon V\to V$ be a linear transformation with $L(V)\subset W$. Denote the restriction of L to W by L_W. Prove that $\det(1-tL)=\det(1-tL_W)$.*

Problem 9 *Let $P(x)$ be a polynomial with real coefficients and with leading coefficient 1. Suppose that $P(0)=-1$ and that $P(x)$ has no complex zeros inside the unit circle. Prove that $P(1)=0$.*

Problem 10 *Let $f_n\colon[0,1]\to[0,\infty)$ be a continuous function, forn $=1,2,\ldots$. Suppose that one has*

$$(*)\qquad f_1(x)\geqslant f_2(x)\geqslant f_3(x)\geqslant\cdots\quad \textit{for all } x\in[0,1].$$

Let $f(x)=\lim_{n\to\infty} f_n(x)$ and $M=\sup_{0\leqslant x\leqslant 1} f(x)$.

1. *Prove that there exists $t\in[0,1]$ with $f(t)=M$.*
2. *Show by example that the conclusion of Part 1 need not hold if instead of $(*)$ we merely know that for each $x\in[0,1]$ there exists n_x such that for all $n\geqslant n_x$ one has $f_n(x)\geqslant f_{n+1}(x)$.*

Problem 11 *Let n be a positive integer, and let $S\subset\mathbb{R}^n$ a finite subset with $0\in S$. Suppose that $\varphi : S\to S$ is a map satisfying*

$$\varphi(0)=0,$$
$$d(\varphi(s),\varphi(t))=d(s,t)\qquad \textit{for all}\quad s,t\in S,$$

where $d(\ ,\ )$ denotes Euclidean metric. Prove that there is a linear mapf $:\mathbb{R}^n\to\mathbb{R}^n$ whose restriction to S is φ.

Problem 12 *Let n be a positive integer. Compute*

$$\int_0^{2\pi}\frac{1-\cos n\theta}{1-\cos\theta}d\theta\,.$$

Problem 13 *Let n be an odd positive integer, and denote by S_n the group of all permutations of $\{1, 2, \ldots, n\}$. Suppose that G is a subgroup of S_n of 2-power order. Prove that there exists $i \in \{1, 2, \ldots, n\}$ such that for all $\sigma \in G$, one has $\sigma(i) = i$.*

Problem 14 *Let $y : \mathbb{R} \to \mathbb{R}$ be a three times differentiable function satisfying the differential equation $y''' - y = 0$. Suppose that $\lim_{x\to\infty} y(x) = 0$. Find real numbers a, b, c, and d, not all zero, such that $ay(0) + y'(0) + cy''(0) = d$.*

Problem 15 *Let $\mathbf{F}$ be a finite field, and suppose that the subfield of $\mathbf{F}$ generated by $\{x^3 \mid x \in \mathbf{F}\}$ is different from $\mathbf{F}$. Show that $\mathbf{F}$ has cardinality 4.*

Problem 16 *Let K be a nonempty compact set in a metric space with distance function d. Suppose that $\varphi\colon K \to K$ satisfies*

$$d(\varphi(x), \varphi(y)) < d(x, y)$$

for all $x \neq y$ in K. Show there exists precisely one point $x \in K$ such that $x = \varphi(x)$.

Problem 17 *Let V be a finite-dimensional vector space over a field $\mathbf{F}$, and let $L : V \to V$ be a linear transformation. Suppose that the characteristic polynomial χ of L is written as $\chi = \chi_1\chi_2$, where χ_1 and χ_2 are two relatively prime polynomials with coefficients in $\mathbf{F}$. Show that V can be written as the direct sum of two subspaces V_1 and V_2 with the property that $\chi_i(L)V_i = 0$ for $i = 1, 2$.*

Problem 18 *Prove that there is no one-to-one conformal mapof the punctured disc $G = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ onto the annulus$A = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$.*