Preliminary Exam - Spring 1994

Problem 1 *Let the collection* $\mathcal{U}$ *of open subsets of* $\mathbb{R}$ *cover the interval* $[0, 1]$. *Prove that there is a positive number* δ *such that any two points* x *and* y *of* $[0, 1]$ *satisfying* $|x - y| < \delta$ *belong together to some member of the cover* $\mathcal{U}$.

Problem 2 *Let* A *be a real* $n \times n$ *matrix. Let* M *denote the maximum of the absolute values of the eigenvalues of* A.

1. *Prove that if* A *is symmetric, then* $\|Ax\| \leqslant M\|x\|$ *for all* x *in* $\mathbb{R}^n$. *(Here,* $\|\cdot\|$ *denotes the Euclidean norm.)*

2. *Prove that the preceding inequality can fail if* A *is not symmetric.*

Problem 3 *Evaluate*

$$\int_{-\pi}^{\pi} \frac{d\theta}{3 - \cos\theta}.$$

Problem 4 *Let* G *be a group having a subgroup* A *of finite index. Prove that there is a normal subgroup* N *of* G *contained in* A *such that* N *is of finite index in* G.

Problem 5

1. *Suppose the functions* $\sin t$ *and* $\sin 2t$ *are both solutions of the differential equation*

$$\sum_{k=0}^{n} c_k \frac{d^k x}{dt^k} = 0 \ ,$$

where $c_0, \ldots, c_n$ *are real constants. What is the smallest possible order of the equation? Write down an equation of minimum order having the given functions as solutions.*

2. *Will the answers to Part 1 be different if the constants* $c_0, \ldots, c_n$ *are allowed to be complex?*

Problem 6 *Prove or disprove: A square complex matrix,* A, *is similar to its transpose,* A^t.

Problem 7 *Let $a_1, a_2, \dots, a_n$ be complex numbers. Prove that there is a point x in $[0,1]$ such that*

$$\left|1 - \sum_{k=1}^{n} a_k e^{2\pi i k x}\right| \geqslant 1.$$

Problem 8 *Find all automorphisms of $\mathbb{Z}[x]$, the ring of polynomials over $\mathbb{Z}$.*

Problem 9

1. *Let U and V be open connected subsets of the complex plane, and let f be an analytic function in U such that $f(U) \subset V$. Assume $f^{-1}(K)$ is compact whenever K is a compact subset of V. Prove that $f(U) = V$.*

2. *Prove that the last equality can fail if* analytic *is replaced by* continuous *in the preceding statement.*

Problem 10 *Let f be a continuous real valued function on $\mathbb{R}$ such that the improper Riemann integral $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges. Define the function g on $\mathbb{R}$ by*

$$g(y) = \int_{-\infty}^{\infty} f(x) \cos(xy)\,dx\,.$$

Prove that g is continuous.

Problem 11 *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a diagonalizable linear transformation. Prove that there is an orthonormal basis for $\mathbb{R}^n$ with respect to which T has an upper-triangular matrix.*

Problem 12 *Let $f = u + iv$ and $g = p + iq$ be analytic functions defined in a neighborhood of the origin in the complex plane. Assume$|g'(0)| < |f'(0)|$. Prove that there is a neighborhood of the origin in which the function $h = f + \overline{g}$ is one-to-one.*

Problem 13 *Let $\mathbf{F}$ be a finite field with q elements. Say that a function $f : \mathbf{F} \to \mathbf{F}$ is a* polynomial function *if there are elements $a_0, a_1, \dots, a_n$ of $\mathbf{F}$ such that $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ for all $x \in \mathbf{F}$. How many polynomial functions are there?*

Problem 14 *Let W be a real 3×3 antisymmetric matrix, i.e.,$W^t = -W$. Let the function*

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

be a real solution of the vector differential equation $\dfrac{dX}{dt} = WX$.

1. *Prove that $\|X(t)\|$, the Euclidean norm of $X(t)$, is independent of t.*
2. *Prove that if v is a vector in the null space of W, then $X(t) \cdot v$ is independent of t.*
3. *Prove that the values $X(t)$ all lie on a fixed circle in $\mathbb{R}^3$.*

Problem 15 *Let α be a number in $(0, 1)$. Prove that any sequence $(x_n)_0^\infty$ of real numbers satisfying the recurrence relation*

$$x_{n+1} = \alpha x_n + (1 - \alpha)x_{n-1}$$

has a limit, and find an expression for the limit in terms of α, x_0, and x_1.

Problem 16 *For which numbers a in $(1, \infty)$ is it true that $x^a \leqslant a^x$ for all x in $(1, \infty)$?*

Problem 17 *Prove that there are at least two nonisomorphic nonabelian groups of order* 30.

Problem 18 *Let u be a real valued harmonic function in the complex plane such that*

$$u(z) \leqslant a\,|\log|z|| + b$$

for all z, where a and b are positive constants. Prove that u is constant.