Preliminary Exam - Spring 1993

Problem 1 *Let* (a_n) *and* (ε_n) *be sequences of positive numbers. Assume that* $\lim_{n\to\infty} \varepsilon_n = 0$ *and that there is a number* k *in* $(0, 1)$ *such that* $a_{n+1} \leqslant ka_n + \varepsilon_n$ *for every* n. *Prove that* $\lim\limits_{n\to\infty} a_n = 0$.

Problem 2 *Let* $A = (a_{ij})$ *be an* $n \times n$ *matrix such that* $\sum\limits_{j=1}^{n} |a_{ij}| < 1$ *for each* i. *Prove that* $I - A$ *is invertible.*

Problem 3 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(1 + x^2)^2} \, dx \, .$$

Problem 4 *Suppose that the group* G *is generated by elements* x *and* y *that satisfy* $x^5y^3 = x^8y^5 = 1$. *Does it follow that* G *is the trivial group?*

Problem 5 *Let* k *be a positive integer. For which values of the real number* c *does the differential equation*

$$\frac{d^2x}{dt^2} - 2c\frac{dx}{dt} + x = 0$$

have a solution satisfying $x(0) = x(2\pi k) = 0$*?*

Problem 6 *Find a list of real matrices, as long as possible, such that*

- *the characteristic polynomial of each matrix is* $(x - 1)^5(x + 1)$,
- *the minimal polynomial of each matrix is* $(x - 1)^2(x + 1)$,
- *no two matrices in the list are similar to each other.*

Problem 7 *Let* $f_n : [0, 4] \to \mathbb{R}$ $(n = 1, 2, \ldots)$ *be continuous functions that are twice continuously differentiable on* $(0, 4)$ *and satisfy*

- $f_n(1) = f_n'(1) = 0$;
- $|f_n'(x)| \leqslant C$ *for all* x *in* $(0, 4)$, *where* C *is a constant, independent of* n.

Prove that the sequence $\{f_n\}$ has a uniformly convergent subsequence.

Problem 8 *Classify up to isomorphism all groups of order 45.*

Problem 9 *Let a be a complex number and ε a positive number. Prove that the function $f(z) = \sin z + \frac{1}{z-a}$ has infinitely many zeros in the strip $|\Im z| < \varepsilon$.*

Problem 10 *Let f be a real valued C^1 function on $[0, \infty)$ such that the improper integral $\int_1^\infty |f'(x)|dx$ converges. Prove that the infinite series $\sum_{n=1}^\infty f(n)$ converges if and only if the integral $\int_1^\infty f(x)dx$ converges.*

Problem 11 *Let P be the vector space of polynomials over $\mathbb{R}$. Let the linear transformation $E : P \to P$ be defined by $Ef = f + f'$, where f' is the derivative of f. Prove that E is invertible.*

Problem 12 *Prove that for any fixed complex number ζ,*

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2\zeta\cos\theta} d\theta = \sum_{n=0}^{\infty}\left(\frac{\zeta^n}{n!}\right)^2.$$

Problem 13 *Prove that no commutative ring with identity has additive group isomorphic to $\mathbb{Q}/\mathbb{Z}$.*

Problem 14 *Prove that every solution $x(t)$ $(t \geqslant 0)$ of the differential equation*

$$\frac{dx}{dt} = x^2 - x^6$$

with $x(0) > 0$ satisfies $\lim_{t\to\infty} x(t) = 1$.

Problem 15 *Let Λ be the set of 2×2 matrices of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

where a and b are elements of a given field $\mathbf{F}$. Prove that Λ, with the usual matrix operations, is a commutative ring with identity. For which of the following fields $\mathbf{F}$ is Λ a field? $\mathbf{F} = \mathbb{Q},\ \mathbb{C},\ \mathbb{Z}_5,\ \mathbb{Z}_7$.

Problem 16 *Prove that $\dfrac{x^2+y^2}{4} \leqslant e^{x+y-2}$ for $x \geqslant 0\,,\ y \geqslant 0\,.$*

Problem 17 *Prove that if G is a group containing no subgroup of index 2, then any subgroup of index 3 in G is a normal subgroup.*

Problem 18 *Let f be an analytic function in the unit disc, $|z| < 1$.*

1. *Prove that there is a sequence (z_n) in the unit disc with $\lim_{n\to\infty} |z_n| = 1$ and $\lim_{n\to\infty} f(z_n)$ exists (finitely).*

2. *Assume f nonconstant. Prove that there are two sequences (z_n) and (w_n) in the disc such that $\lim_{n\to\infty} |z_n| = \lim_{n\to\infty} |w_n| = 1$, and such that both limits $\lim_{n\to\infty} f(z_n)$ and $\lim_{n\to\infty} f(w_n)$ exist (finitely) and are not equal.*