Preliminary Exam - Spring 1992

Problem 1 1. *Prove that every finitely generated subgroup of $\mathbb{Q}$, the additive group of rational numbers, is cyclic.*

2. *Does the same conclusion hold for finitely generated subgroups of $\mathbb{Q}/\mathbb{Z}$, where $\mathbb{Z}$ is the group of integers?*

Problem 2 *Find a square root of the matrix*

$$\begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{pmatrix}.$$

How many square roots does this matrix have?

Problem 3 *Let f be an analytic function in the connected open subset G of the complex plane. Assume that for each point z in G, there is a positive integer n such that the n^{th} derivative of f vanishes at z. Prove that f is a polynomial.*

Problem 4 *Show that every infinite closed subset of $\mathbb{R}^n$ is the closure of a countable set.*

Problem 5 *Let f be a differentiable function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Assume that there is a differentiable function g from $\mathbb{R}^n$ to $\mathbb{R}$ having no critical points such that $g \circ f$ vanishes identically. Prove that the Jacobian determinant of f vanishes identically.*

Problem 6 *Let p be a prime integer, $p \equiv 3 \pmod 4$, and let $\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$. If $x^4 + 1$ factors into a product $g(x)h(x)$ of two quadratic polynomials in $\mathbf{F}_p[x]$, prove that $g(x)$ and $h(x)$ are both irreducible over $\mathbf{F}_p$.*

Problem 7 *Let $a_1, a_2, \ldots, a_{10}$ be integers with $1 \leqslant a_i \leqslant 25$, for $1 \leqslant i \leqslant 10$. Prove that there exist integers $n_1, n_2, \ldots, n_{10}$, not all zero, such that*

$$\prod_{i=1}^{10} a_i^{n_i} = 1.$$

Problem 8 *Evaluate the integral*

$$I = \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}\, dx \,.$$

Problem 9 *Let p be a nonconstant polynomial with real coefficients and only real roots. Prove that for each real number r, the polynomial $p - rp'$ has only real roots.*

Problem 10 *Let A denote the matrix*

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For which positive integers n is there a complex 4×4 matrix X such that $X^n = A$?

Problem 11 *Find a Laurent series that converges in the annulus*$1 < |z| < 2$ *to a branch of the function* $\log\left(\frac{z(2-z)}{1-z}\right)$.

Problem 12 *Let A be a real symmetric $n \times n$ matrix with nonnegative entries. Prove that A has an eigenvector with nonnegative entries.*

Problem 13 *Let f be a one-to-one C^1 map of $\mathbb{R}^3$ into $\mathbb{R}^3$, and let J denote its Jacobian determinant. Prove that if x_0 is any point of $\mathbb{R}^3$ and $Q_r(x_0)$ denotes the cube with center x_0, side length r, and edges parallel to the coordinate axes, then*

$$|J(x_0)| = \lim_{r \to 0} r^{-3} \mathrm{vol}\,(f(Q_r(x_0))) \leqslant \limsup_{x \to x_0} \frac{\|f(x) - f(x_0)\|^3}{\|x - x_0\|^3}.$$

Here, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$.

Problem 14

1. *Prove that $\alpha = \sqrt{5}+\sqrt{7}$ is algebraic over $\mathbb{Q}$, by explicitly finding a polynomial $f(x)$ in $\mathbb{Q}[x]$ of degree 4 having α as a root.*
2. *Prove that $f(x)$ is irreducible over $\mathbb{Q}$.*

Problem 15 *Let* S_{999} *denote the group of permutations of* 999 *objects, and let* $G \subset S_{999}$ *be an abelian subgroup of order* 1111. *Prove that there exists* $i \in \{1, \ldots, 999\}$ *such that for all* $\sigma \in G$, *one has* $\sigma(i) = i$.

Problem 16 *Let* $x_0 = 1$ *and*

$$x_{n+1} = \frac{3 + 2x_n}{3 + x_n}\,, \qquad n \geqslant 0\,.$$

Prove that $x_\infty = \lim_{n\to\infty} x_n$ *exists, and find its value.*

Problem 17 *For which positive numbers* a *and* b, *with* $a > 1$, *does the equation* $\log_a x = x^b$ *have a positive solution for* x*?*

Problem 18 *Let the function* f *be analytic in the entire complex plane, real valued on the real axis, and of positive imaginary part in the upper half-plane. Prove* $f'(x) > 0$ *for* x *real.*