# Preliminary Exam - Spring 1991

**Problem 1** *List, to within isomorphism, all the finite groups whose orders do not exceed* 5*.*

**Problem 2** *Let $f$ be a continuous complex valued function on* $[0,1]$*, and define the function $g$ by*

$$g(z) = \int_0^1 f(t)e^{tz}\,dt \qquad (z \in \mathbb{C}\,).$$

*Prove that $g$ is analytic in the entire complex plane.*

**Problem 3** *For $n$ a positive integer, let $d(n)$ denote the number of positive integers that divide $n$. Prove that $d(n)$ is odd if and only if $n$ is a perfect square.*

**Problem 4** *Let $p$ be a prime number and $R$ a ring with identity containing $p^2$ elements. Prove that $R$ is commutative.*

**Problem 5** *Let $A = (a_{ij})_{i,j=1}^r$ be a square matrix with integer entries.*

1. *Prove that if an integer $n$ is an eigenvalue of $A$, then $n$ is a divisor of* $\det A$*, the determinant of $A$.*

2. *Suppose that $n$ is an integer and that each row of $A$ has sum $n$:*

$$\sum_{j=1}^{r} a_{ij} = n, \qquad 1 \leqslant i \leqslant r.$$

   *Prove that $n$ is a divisor of* $\det A$*.*

**Problem 6** *Let the vector field $F$ in $\mathbb{R}^3$ have the form*

$$F(r) = g(\|r\|)r \qquad (r \neq (0,0,0)),$$

*where $g$ is a real valued smooth function on $(0,\infty)$ and $\|\cdot\|$ denotes the Euclidean norm. ($F$ is undefined at $(0,0,0)$.) Prove that*

$$\int_C F \cdot ds = 0$$

*for any smooth closed path $C$ in $\mathbb{R}^3$ that does not pass through the origin.*

**Problem 7** *Let the function $f$ be analytic in the unit disc, with $|f(z)| \leqslant 1$ and $f(0) = 0$. Assume that there is a number $r$ in $(0, 1)$ such that $f(r) = f(-r) = 0$. Prove that*

$$|f(z)| \leqslant |z| \left| \frac{z^2 - r^2}{1 - r^2 z^2} \right| .$$

**Problem 8** *Let $T$ be a real, symmetric, $n \times n$,* tridiagonal *matrix:*

$$T = \begin{pmatrix} a_1 & b_1 & 0 & 0 & \cdots & 0 & 0 \\ b_1 & a_2 & b_2 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & a_3 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}$$

*(All entries not on the main diagonal or the diagonals just above and below the main one are zero.) Assume $b_j \neq 0$ for all $j$.*

*Prove:*

1. $\operatorname{rank} T \geqslant n - 1$.
2. *$T$ has $n$ distinct eigenvalues.*

**Problem 9** *Let $f$ be a continuous function from the ball $B_n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ into itself. (Here, $\|\cdot\|$ denotes the Euclidean norm.) Assume $\|f(x)\| < \|x\|$ for all nonzero $x \in B_n$. Let $x_0$ be a nonzero point of $B_n$, and define the sequence $(x_k)$ by setting $x_k = f(x_{k-1})$. Prove that $\lim x_k = 0$.*

**Problem 10** *Prove that $\mathbb{Q}$ , the additive group of rational numbers, cannot be written as the direct sum of two nontrivial subgroups.*
*Note: See also Problems* **??** *and* **??**.

**Problem 11** *For which real numbers $x$ does the infinite series*

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^x}$$

*converge?*

**Problem 12** *Let $A$ be the set of positive integers that do not contain the digit 9 in their decimal expansions. Prove that*

$$\sum_{a \in A} \frac{1}{a} < \infty;$$

*that is, $A$ defines a convergent subseries of the harmonic series.*

**Problem 13** *Prove that*

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{\sin x}{x - 3i}\, dx$$

*exists and find its value.*

**Problem 14** *Let $x(t)$ be a nontrivial solution to the system*

$$\frac{dx}{dt} = Ax,$$

*where*

$$A = \begin{pmatrix} 1 & 6 & 1 \\ -4 & 4 & 11 \\ -3 & -9 & 8 \end{pmatrix}.$$

*Prove that $\|x(t)\|$ is an increasing function of $t$. (Here, $\|\cdot\|$ denotes the Euclidean norm.)*

**Problem 15** *Let $G$ be a finite nontrivial group with the property that for any two elements $a$ and $b$ in $G$ different from the identity, there is an element $c$ in $G$ such that $b = c^{-1}ac$. Prove that $G$ has order 2.*

**Problem 16** *Let $A$ be a linear transformation on an $n$-dimensional vector space over $\mathbb{C}$ with characteristic polynomial $(x-1)^n$. Prove that $A$ is similar to $A^{-1}$.*

**Problem 17** *Let the function $f$ be analytic in the punctured disc$0 < |z| < r_0$, with Laurent series*

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n.$$

*Assume there is a positive number $M$ such that*

$$r^4 \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta < M, \qquad 0 < r < r_0.$$

*Prove that $c_n = 0$ for $n < -2$.*

**Problem 18** *Let the real valued function $f$ be defined in an open interval about the point $a$ on the real line and be differentiable at $a$. Prove that if $(x_n)$ is an increasing sequence and $(y_n)$ is a decreasing sequence in the domain of $f$, and both sequences converge to $a$, then*

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$