Preliminary Exam - Spring 1990

Problem 1 *Let $y : \mathbb{R} \to \mathbb{R}$ be a C^∞ function that satisfies the differential equation*

$$y'' + y' - y = 0$$

for $x \in [0, L]$, where L is a positive real number. Suppose that $y(0) = y(L) = 0$. Prove that $y \equiv 0$ on $[0, L]$.

Problem 2 *Let A be a complex $n \times n$ matrix that has* finite order*; that is, $A^k = I$ for some positive integer k. Prove that A is diagonalizable.*

Problem 3 *Let $c_0, c_1, \ldots, c_{n-1}$ be complex numbers. Prove that all the zeros of the polynomial*

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$$

lie in the open disc with center 0 and radius

$$\sqrt{1 + |c_{n-1}|^2 + \cdots + |c_1|^2 + |c_0|^2}\,.$$

Problem 4 *Let R be a commutative ring with 1, and R^* be its group of units. Suppose that the additive group of R is generated by $\{u^2 \mid u \in R^*\}$. Prove that R has, at most, one ideal $\mathfrak{I}$ for which $R/\mathfrak{I}$ has cardinality 3.*

Problem 5 *Suppose $x_1, x_2, x_3, \ldots$ is a sequence of nonnegative real numbers satisfying*

$$x_{n+1} \leqslant x_n + \frac{1}{n^2}$$

for all $n \geqslant 1$. Prove that $\lim_{n\to\infty} x_n$ exists.

Problem 6 *Give an example of a continuous function $v : \mathbb{R} \to \mathbb{R}^3$ with the property that $v(t_1)$, $v(t_2)$, and $v(t_3)$ form a basis for $\mathbb{R}^3$ whenever t_1, t_2, and t_3 are distinct points of $\mathbb{R}$.*

Problem 7 *Let a be a positive real number. Evaluate the improper integral*

$$\int_0^\infty \frac{\sin x}{x(x^2 + a^2)}\,dx\,.$$

Problem 8 *Let $\mathbb{C}^*$ be the multiplicative group of nonzero complex numbers. Suppose that H is a subgroup of finite index of $\mathbb{C}^*$. Prove that $H = \mathbb{C}^*$.*

Problem 9 *Let the real valued function f on $[0,1]$ have the following two properties:*

- *If $[a,b] \subset [0,1]$, then $f([a,b])$ contains the interval with endpoints $f(a)$ and $f(b)$ (i.e., f has the Intermediate Value Property).*
- *For each $c \in \mathbb{R}$, the set $f^{-1}(c)$ is closed.*

Prove that f is continuous.

Problem 10 *Show that there are at least two nonisomorphic nonabelian groups of order 24, of order 30 and order 40.*

Problem 11 *Let the function f be analytic and bounded in the complex half-plane $\Re z > 0$. Prove that for any positive real number c, the function f is uniformly continuous in the half-plane $\Re z > c$.*

Problem 12 *Let n be a positive integer, and let $A = (a_{ij})_{i,j=1}^n$ be the $n \times n$ matrix with $a_{ii} = 2$, $a_{i\,i\pm 1} = -1$, and $a_{ij} = 0$ otherwise; that is,*

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

Prove that every eigenvalue of A is a positive real number.

Problem 13 *Let f be an infinitely differentiable function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for some positive integer n,*

$$f(1) = f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0.$$

Prove that $f^{(n+1)}(x) = 0$ for some x in $(0,1)$.

Problem 14 *Let A and B be subspaces of a finite-dimensional vector space V such that $A + B = V$. Write $n = \dim V$, $a = \dim A$, and $b = \dim B$. Let S be the set of those endomorphisms f of V for which $f(A) \subset A$ and $f(B) \subset B$. Prove that S is a subspace of the set of all endomorphisms of V, and express the dimension of S in terms of n, a, and b.*

Problem 15 *Find a one-to-one conformal map of the semidisc*

$$\left\{ z \in \mathbb{C} \mid \Im z > 0, \left| z - \frac{1}{2} \right| < \frac{1}{2} \right\}$$

onto the upper half-plane.

Problem 16 *Determine the greatest common divisor of the elements of the set $\{n^{13} - n \mid n \in \mathbb{Z}\}$.*

Problem 17 *Let f be a differentiable function on $[0, 1]$ and let*

$$\sup_{0<x<1} |f'(x)| = M < \infty.$$

Let n be a positive integer. Prove that

$$\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)\, dx \right| \leqslant \frac{M}{2n}.$$

Problem 18 *Let $z_1, z_2, \ldots, z_n$ be complex numbers. Prove that there exists a subset $J \subset \{1, 2, \ldots, n\}$ such that*

$$\left| \sum_{j \in J} z_j \right| \geqslant \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j|.$$