Preliminary Exam - Spring 1989

Problem 1 *Let $a_1, a_2, \ldots$ be positive numbers such that*

$$\sum_{n=1}^{\infty} a_n < \infty\,.$$

Prove that there are positive numbers $c_1, c_2, \ldots$ such that

$$\lim_{n\to\infty} c_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} c_n a_n < \infty\,.$$

Problem 2 *Let $\mathbf{F}$ be a field, n and m positive integers, and A an $n \times n$ matrix with entries in $\mathbf{F}$ such that $A^m = 0$. Prove that $A^n = 0$.*

Problem 3 *Let f be a continuous real valued function defined on $[0,1] \times [0,1]$. Let the function g on $[0,1]$ be defined by*

$$g(x) = \max\{f(x,y) \mid y \in [0,1]\}\,.$$

Prove that g is continuous.

Problem 4 *Prove that if $1 < \lambda < \infty$, the function*

$$f_\lambda(z) = z + \lambda - e^z$$

has only one zero in the half-plane $\Re z < 0$, and this zero is on the real axis.

Problem 5 *Let G be a group whose order is twice an odd number. For g in G, let λ_g denote the permutation of G given by $\lambda_g(x) = gx$ for $x \in G$.*

1. *Let g be in G. Prove that the permutation λ_g is even if and only if the order of g is odd.*

2. *Let $N = \{g \in G \mid \mathrm{order}(g) \text{ is odd}\}$. Prove that N is a normal subgroup of G of index 2.*

Problem 6 *Let S be the subspace of $M_{n\times n}$ (the vector space of all real $n \times n$ matrices) generated by all matrices of the form $AB - BA$ with A and B in $M_{n\times n}$. Prove that* $\dim(S) = n^2 - 1$.

Problem 7 *Let*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Find the general solution of the matrix differential equation $\dfrac{dX}{dt} = AXB$ for the unknown 4×4 matrix function $X(t)$.

Problem 8 *Let f be an analytic function mapping the open unit disc, D, into itself. Assume that there are two points a and b in D, with $a \neq b$, such that $f(a) = a$ and $f(b) = b$. Prove that f is the identity function ($f(z) = z$).*

Problem 9 *For G a group and H a subgroup, let $C(G, H)$ denote the collection of left cosets of H in G. Prove that if H and K are two subgroups of G of infinite index, then G is not a finite union of cosets from $C(G, H) \cup C(G, K)$.*

Problem 10 *Let the real $2n \times 2n$ matrix X have the form*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, and D are $n \times n$ matrices that commute with one another. Prove that X is invertible if and only if $AD - BC$ is invertible.

Problem 11 *Let R be a ring with at least two elements. Suppose that for each nonzero a in R there is a unique b in R (depending on a) with $aba = a$. Show that R is a division ring.*

Problem 12 *Evaluate*

$$\int_C (2z - 1)e^{z/(z-1)}\, dz$$

where C is the circle $|z| = 2$ with counterclockwise orientation.

Problem 13 *Let D_n be the dihedral group, the group of rigid motions of a regular n-gon ($n \geqslant 3$). (It is a noncommutative group of order $2n$.) Determine its center $Z = \{c \in D_n \mid cx = xc \text{ for all } x \in D_n\}$.*

Problem 14 *Suppose f is a continuously differentiable map of $\mathbb{R}^2$ into $\mathbb{R}^2$. Assume that f has only finitely many singular points, and that for each positive number M, the set $\{z \in \mathbb{R}^2 \mid |f(z)| \leqslant M\}$ is bounded. Prove that f maps $\mathbb{R}^2$ onto $\mathbb{R}^2$.*

Problem 15 *Let $B = (b_{ij})_{i,j=1}^{20}$ be a real 20×20 matrix such that*

$$b_{ii} = 0 \quad for \quad 1 \leqslant i \leqslant 20,$$

$$b_{ij} \in \{1, -1\} \quad for \quad 1 \leqslant i, j \leqslant 20, \;\; i \neq j.$$

Prove that B is nonsingular.

Problem 16 *Let f and g be entire functions such that $\lim_{z\to\infty} f(g(z)) = \infty$. Prove that f and g are polynomials.*

Problem 17

1. *Let R be a commutative ring with identity containing an element a with $a^3 = a + 1$. Further, let $\mathfrak{I}$ be an ideal of R ofindex < 5 in R. Prove that $\mathfrak{I} = R$.*
2. *Show that there exists a commutative ring with identity that has an element a with $a^3 = a + 1$ and that contains an ideal of index 5.*

Note: The term index *is used here exactly as in group theory; namely the index of $\mathfrak{I}$ in R means the order of $R/\mathfrak{I}$.*

Problem 18 *Let f be a real valued function on $\mathbb{R}^2$ with the following properties:*

1. *For each y_0 in $\mathbb{R}$, the function $x \mapsto f(x, y_0)$ is continuous.*
2. *For each x_0 in $\mathbb{R}$, the function $y \mapsto f(x_0, y)$ is continuous.*
3. *$f(K)$ is compact whenever K is a compact subset of $\mathbb{R}^2$.*

Prove that f is continuous.