Preliminary Exam - Spring 1988

Problem 1 *Suppose that* $f(x)$, $-\infty < x < \infty$, *is a continuous real valued function, that* $f'(x)$ *exists for* $x \neq 0$, *and that* $\lim_{x\to 0} f'(x)$ *exists. Prove that* $f'(0)$ *exists.*

Problem 2 *Determine the last digit of*

$$23^{23^{23^{23}}}$$

in the decimal system.

Problem 3 *If a finite homogeneous system of linear equations with rational coefficients has a nontrivial complex solution, need it have a nontrivial rational solution? Give a proof or a counterexample.*

Problem 4 *True or false: A function* $f(z)$ *analytic on* $|z-a| < r$ *and continuous on* $|z-a| \leqslant r$ *extends, for some* $\delta > 0$, *to a function analytic on* $|z-a| < r+\delta$? *Give a proof or a counterexample.*

Problem 5 *Let* D *be a group of order* $2n$, *where* n *is odd, with a subgroup* H *of order* n *satisfying* $xhx^{-1} = h^{-1}$ *for all* h *in* H *and all* x *in* $D \setminus H$. *Prove that* H *is commutative and that every element of* $D \setminus H$ *is of order* 2.

Problem 6 *Prove or disprove: There is a real* $n \times n$ *matrix* A *such that*

$$A^2 + 2A + 5I = 0$$

if and only if n *is even.*

Problem 7 *Let* R *be a commutative ring with identity element and* $a \in R$. *Let* n *and* m *be positive integers, and write* $d = \gcd\{n, m\}$. *Prove that the ideal of* R *generated by* $a^n - 1$ *and* $a^m - 1$ *is the same as the ideal generated by* $a^d - 1$.

Problem 8 *For* $a > 1$ *and* $n = 0, 1, 2, \ldots$, *evaluate the integrals*

$$C_n(a) = \int_{-\pi}^{\pi} \frac{\cos n\theta}{a - \cos\theta}\, d\theta\,, \qquad S_n(a) = \int_{-\pi}^{\pi} \frac{\sin n\theta}{a - \cos\theta}\, d\theta\,.$$

Problem 9 *Prove that the integrals*

$$\int_0^\infty \cos x^2\,dx \quad \text{and} \quad \int_0^\infty \sin x^2\,dx$$

converge.

Problem 10

1. *Let G be an open connected subset of the complex plane, f an analytic function in G, not identically 0, and n a positive integer. Assume that f has an analytic n^{th} root in G; that is, there is an analytic function g in G such that $g^n = f$. Prove that f has exactly n analytic n^{th} roots in G.*

2. *Give an example of a continuous real valued function on $[0,1]$ that has more than two continuous square roots on $[0,1]$.*

Problem 11 *Let S_9 denote the group of permutations of nine objects and let A_9 be the subgroup consisting of all even permutations. Denote by $1 \in S_9$ the identity permutation. Determine the minimum of all positive integers m such that every $\sigma \in S_9$ satisfies $\sigma^m = 1$. Determine also the minimum of all positive integers m such that every $\sigma \in A_9$ satisfies $\sigma^m = 1$.*

Problem 12 *For each real value of the parameter t, determine the number of real roots, counting multiplicities, of the cubic polynomial$p_t(x) = (1 + t^2)x^3 - 3t^3x + t^4$.*

Problem 13 *Find all groups G such that every automorphism of G is trivial.*

Problem 14 *Let the function f be analytic in the open unit disc of the complex plane and real valued on the radii $[0,1)$ and $[0, e^{i\pi\sqrt{2}})$. Prove that f is constant.*

Problem 15 *Compute A^{10} for the matrix*

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Problem 16 *Let X be a set and V a real vector space of real valued functions on X of dimension n, $0 < n < \infty$. Prove that there are n points $x_1, x_2, \ldots, x_n$ in X such that the map $f \mapsto (f(x_1), \ldots, f(x_n))$ of V to $\mathbb{R}^n$ is an isomorphism.*

Problem 17 1. *Let f be an analytic function that maps the open unit disc, $\mathbb{D}$, into itself and vanishes at the origin. Prove that* $|f(z)+f(-z)| \leqslant 2|z|^2$ *in* $\mathbb{D}$.

2. *Prove that the inequality in Part 1 is strict, except at the origin, unless f has the form $f(z) = \lambda z^2$ with λ a constant of absolute value one.*

Problem 18 *For which positive integers n is there a 2×2 matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries and order n; that is, $A^n = I$ but $A^k \neq I$ for $0 < k < n$? Note: See also Problem **??**.

Problem 19 *Show that one can represent the set of nonnegative integers, $\mathbb{Z}_+$, as the union of two disjoint subsets N_1 and N_2 ($N_1 \cap N_2 = \emptyset$, $N_1 \cup N_2 = \mathbb{Z}_+$) such that neither N_1 nor N_2 contains an infinite arithmetic progression.*

Problem 20 *Does there exist a continuous real valued function $f(x)$, $0 \leqslant x \leqslant 1$, such that*

$$\int_0^1 x f(x)\,dx = 1 \quad \text{and} \quad \int_0^1 x^n f(x)\,dx = 0$$

for $n = 0, 2, 3, 4, \ldots$? Give an example or a proof that no such f exists.