Problem 1  A standard theorem states that a continuous real valued function on a compact set is bounded. Prove the converse: If $K$ is a subset of $\mathbb{R}^n$ and if every continuous real valued function on $K$ is bounded, then $K$ is compact.

Problem 2  Let the transformation $T$ from the subset $U = \{(u, v) \mid u > v\}$ of $\mathbb{R}^2$ into $\mathbb{R}^2$ be defined by $T(u, v) = (u + v, u^2 + v^2)$.

1. Prove that $T$ is locally one-to-one.

2. Determine the range of $T$, and show that $T$ is globally one-to-one.

Problem 3  Let $f$ be a complex valued function in the open unit disc, $\mathbb{D}$, of the complex plane such that the functions $g = f^2$ and $h = f^3$ are both analytic. Prove that $f$ is analytic in $\mathbb{D}$.

Problem 4  Let $F$ be a finite field with $q$ elements and let $x$ be an indeterminate. For $f$ a polynomial in $F[x]$, let $\varphi_f$ denote the corresponding function of $F$ into $F$, defined by $\varphi_f(a) = f(a)$, $(a \in F)$. Prove that if $\varphi$ is any function of $F$ into $F$, then there is an $f$ in $F[x]$ such that $\varphi = \varphi_f$. Prove that $f$ is uniquely determined by $\varphi$ to within addition of a multiple of $x^q - x$.

Problem 5  Let $f$ be a continuous real valued function on $\mathbb{R}$ satisfying

$$|f(x)| \leq \frac{C}{1 + x^2},$$

where $C$ is a positive constant. Define the function $F$ on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + n).$$

1. Prove that $F$ is continuous and periodic with period 1.

2. Prove that if $G$ is continuous and periodic with period 1, then

$$\int_{0}^{1} F(x)G(x) \, dx = \int_{-\infty}^{\infty} f(x)G(x) \, dx.$$
Problem 6 Let \( f \) be an analytic function in the open unit disc of the complex plane such that \(|f(z)| \leq C/(1-|z|)\) for all \( z \) in the disc, where \( C \) is a positive constant. Prove that \(|f'(z)| \leq 4C/(1-|z|)^2\).

Problem 7 Let \( p, q \) and \( r \) be continuous real valued functions on \( \mathbb{R} \), with \( p > 0 \). Prove that the differential equation
\[
p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0
\]
is equivalent to (i.e., has exactly the same solutions as) a differential equation of the form
\[
(a(t)x'(t))' + b(t)x(t) = 0,
\]
where \( a \) is continuously differentiable and \( b \) is continuous.

Problem 8 Prove that if the nonconstant polynomial \( p(z) \), with complex coefficients, has all of its roots in the half-plane \( \Re z > 0 \), then all of the roots of its derivative are in the same half-plane.

Problem 9 Let \( A \) be an \( m \times n \) matrix with rational entries and \( b \) an \( m \)-dimensional column vector with rational entries. Prove or disprove: If the equation \( Ax = b \) has a solution \( x \) in \( \mathbb{C}^n \), then it has a solution with \( x \) in \( \mathbb{Q}^n \).

Problem 10 Prove that any finite group of order \( n \) is isomorphic to a subgroup of \( \text{O}(n) \), the group of \( n \times n \) orthogonal real matrices.

Problem 11 Show that the equation \( ae^{x} = 1+x+x^2/2 \), where \( a \) is a positive constant, has exactly one real root.

Problem 12 Evaluate the integral
\[
I = \int_0^{1/2} \frac{\sin x}{x} \, dx
\]
to an accuracy of two decimal places; that is, find a number \( I^* \) such that \(|I - I^*| < 0.005\).

Problem 13 Let \( f \) be a real valued \( C^1 \) function defined in the punctured plane \( \mathbb{R}^2 \setminus \{(0,0)\} \). Assume that the partial derivatives \( \partial f/\partial x \) and \( \partial f/\partial y \) are uniformly bounded; that is, there exists a positive constant \( M \) such that \(|\partial f/\partial x| \leq M \) and \(|\partial f/\partial y| \leq M \) for all points \((x, y) \neq (0, 0)\). Prove that
\[
\lim_{(x,y) \to (0,0)} f(x, y)
\]
exists.
Problem 14  1. Show that, to within isomorphism, there is just one non-cyclic group $G$ of order 4.

2. Show that the group of automorphisms of $G$ is isomorphic to the permutation group $S_3$.

Problem 15  Prove or disprove: If the function $f$ is analytic in the entire complex plane, and if $f$ maps every unbounded sequence to an unbounded sequence, then $f$ is a polynomial.

Problem 16  Let $F$ be a uniformly bounded, equicontinuous family of real valued functions on the metric space $(X, d)$. Prove that the function

$$g(x) = \sup\{f(x) \mid f \in F\}$$

is continuous.

Problem 17  Let $V$ be a finite-dimensional linear subspace of $C^\infty(\mathbb{R})$ (the space of complex valued, infinitely differentiable functions). Assume that $V$ is closed under $D$, the operator of differentiation (i.e., $f \in V \Rightarrow Df \in V$). Prove that there is a constant coefficient differential operator

$$L = \sum_{k=0}^{n} a_k D^k$$

such that $V$ consists of all solutions of the differential equation $Lf = 0$.

Problem 18  Let $A$ and $B$ be two diagonalizable $n \times n$ complex matrices such that $AB = BA$. Prove that there is a basis for $\mathbb{C}^n$ that simultaneously diagonalizes $A$ and $B$.

Problem 19  Let $F$ be a field. Prove that every finite subgroup of the multiplicative group of nonzero elements of $F$ is cyclic.

Problem 20  Evaluate

$$I = \int_{0}^{\pi} \frac{\cos 4\theta}{1 + \cos^2 \theta} \, d\theta.$$