Preliminary Exam - Spring 1987

Problem 1 *A standard theorem states that a continuous real valued function on a compact set is bounded. Prove the converse: If K is a subset of $\mathbb{R}^n$ and if every continuous real valued function on K is bounded, then K is compact.*

Problem 2 *Let the transformation T from the subset$U = \{(u, v) \mid u > v\}$ of $\mathbb{R}^2$ into $\mathbb{R}^2$ be defined by $T(u, v) = (u + v, u^2 + v^2)$.*

1. *Prove that T is locally one-to-one.*
2. *Determine the range of T, and show that T is globally one-to-one.*

Problem 3 *Let f be a complex valued function in the open unit disc, $\mathbb{D}$, of the complex plane such that the functions $g = f^2$ and $h = f^3$ are both analytic. Prove that f is analytic in $\mathbb{D}$.*

Problem 4 *Let $\mathbf{F}$ be a finite field with q elements and let x be an indeterminate. For f a polynomial in $\mathbf{F}[x]$, let φ_f denote the corresponding function of $\mathbf{F}$ into $\mathbf{F}$, defined by $\varphi_f(a) = f(a)$, $(a \in \mathbf{F})$. Prove that if φ is any function of $\mathbf{F}$ into $\mathbf{F}$, then there is an f in $\mathbf{F}[x]$ such that $\varphi = \varphi_f$. Prove that f is uniquely determined by φ to within addition of a multiple of $x^q - x$.*

Problem 5 *Let f be a continuous real valued function on $\mathbb{R}$ satisfying*

$$|f(x)| \leqslant \frac{C}{1 + x^2},$$

where C is a positive constant. Define the function F on $\mathbb{R}$ by

$$F(x) = \sum_{n=-\infty}^{\infty} f(x + n)\,.$$

1. *Prove that F is continuous and periodic with period 1.*
2. *Prove that if G is continuous and periodic with period 1, then*

$$\int_0^1 F(x)G(x)\,dx = \int_{-\infty}^{\infty} f(x)G(x)\,dx\,.$$

Problem 6 *Let f be an analytic function in the open unit disc of the complex plane such that $|f(z)| \leqslant C/(1-|z|)$ for all z in the disc, where C is a positive constant. Prove that $|f'(z)| \leqslant 4C/(1-|z|)^2$.*

Problem 7 *Let p, q and r be continuous real valued functions on $\mathbb{R}$, with $p > 0$. Prove that the differential equation*

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$

is equivalent to (i.e., has exactly the same solutions as) a differential equation of the form

$$(a(t)x'(t))' + b(t)x(t) = 0,$$

where a is continuously differentiable and b is continuous.

Problem 8 *Prove that if the nonconstant polynomial $p(z)$, with complex coefficients, has all of its roots in the half-plane $\Re z > 0$, then all of the roots of its derivative are in the same half-plane.*

Problem 9 *Let A be an $m \times n$ matrix with rational entries and b an m-dimensional column vector with rational entries. Prove or disprove: If the equation $Ax = b$ has a solution x in $\mathbb{C}^n$, then it has a solution with x in $\mathbb{Q}^n$.*

Problem 10 *Prove that any finite group of order n is isomorphic to a subgroup of $\mathbb{O}(n)$, the group of $n \times n$ orthogonal real matrices.*

Problem 11 *Show that the equation $ae^x = 1+x+x^2/2$, where a is a positive constant, has exactly one real root.*

Problem 12 *Evaluate the integral*

$$I = \int_0^{1/2} \frac{\sin x}{x}\, dx$$

to an accuracy of two decimal places; that is, find a number I^* such that $|I - I^*| < 0.005$.

Problem 13 *Let f be a real valued C^1 function defined in the punctured plane $\mathbb{R}^2 \setminus \{(0,0)\}$. Assume that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are* uniformly bounded*; that is, there exists a positive constant M such that $|\partial f/\partial x| \leqslant M$ and $|\partial f/\partial y| \leqslant M$ for all points $(x, y) \neq (0,0)$. Prove that*

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exists.

Problem 14 1. *Show that, to within isomorphism, there is just one non-cyclic group G of order* 4.

2. *Show that the group of automorphisms of G is isomorphic to the permutation group S_3.*

Problem 15 *Prove or disprove: If the function f is analytic in the entire complex plane, and if f maps every unbounded sequence to an unbounded sequence, then f is a polynomial.*

Problem 16 *Let $\mathcal{F}$ be a uniformly bounded, equicontinuous family of real valued functions on the metric space (X, d). Prove that the function*

$$g(x) = \sup\{f(x) \mid f \in \mathcal{F}\}$$

is continuous.

Problem 17 *Let V be a finite-dimensional linear subspace of $C^\infty(\mathbb{R})$ (the space of complex valued, infinitely differentiable functions). Assume that V is closed under D, the operator of differentiation (i.e., $f \in V \Rightarrow Df \in V$). Prove that there is a constant coefficient differential operator*

$$L = \sum_{k=0}^{n} a_k D^k$$

such that V consists of all solutions of the differential equation $Lf = 0$.

Problem 18 *Let A and B be two diagonalizable $n \times n$ complex matrices such that $\mathrm{AB} = \mathrm{BA}$. Prove that there is a basis for $\mathbb{C}^n$ that simultaneously diagonalizes A and B.*

Problem 19 *Let $\mathbf{F}$ be a field. Prove that every finite subgroup of the multiplicative group of nonzero elements of $\mathbf{F}$ is cyclic.*

Problem 20 *Evaluate*

$$I = \int_0^\pi \frac{\cos 4\theta}{1 + \cos^2\theta}\, d\theta\,.$$