Preliminary Exam - Spring 1986

Problem 1 *Let $e = (a, b, c)$ be a unit vector in $\mathbb{R}^3$ and let T be the linear transformation on $\mathbb{R}^3$ of rotation by $180°$ about e. Find the matrix for T with respect to the standard basis.*

Problem 2 *Let f be a continuous real valued function on $\mathbb{R}$ such that*

$$f(x) = f(x+1) = f\left(x + \sqrt{2}\right)$$

for all x. Prove that f is constant.

Problem 3 *Let C be a simple closed contour enclosing the points $0, 1, 2, \ldots, k$ in the complex plane, with positive orientation. Evaluate the integrals*

$$I_k = \int_C \frac{dz}{z(z-1)\cdots(z-k)}, \qquad k = 0, 1, \ldots,$$

$$J_k = \int_C \frac{(z-1)\cdots(z-k)}{z}\, dz, \qquad k = 0, 1, \ldots.$$

Problem 4 *Let f be a positive differentiable function on $(0, \infty)$. Prove that*

$$\lim_{\delta \to 0} \left(\frac{f(x + \delta x)}{f(x)} \right)^{1/\delta}$$

exists (finitely) and is nonzero for each x.

Problem 5 *Prove that there exists only one automorphism of the field of real numbers; namely the identity automorphism.*

Problem 6 *Let V be a finite-dimensional vector space and A and B two linear transformations of V into itself such that $A^2 = B^2 = 0$ and$AB + BA = I$.*

1. *Prove that if N_A and N_B are the respective null spaces of A and B, then $N_A = AN_B$, $N_B = BN_A$, and $V = N_A \oplus N_B$.*

2. *Prove that the dimension of V is even.*

3. *Prove that if the dimension of V is 2, then V has a basis with respect to which A and B are represented by the matrices*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Problem 7 *For λ a real number, find all solutions of the integral equations*

$$\varphi(x) = e^x + \lambda \int_0^x e^{(x-y)} \varphi(y)\, dy, \qquad 0 \leqslant x \leqslant 1,$$

$$\psi(x) = e^x + \lambda \int_0^1 e^{(x-y)} \psi(y)\, dy, \qquad 0 \leqslant x \leqslant 1.$$

Problem 8 *Let the 3×3 matrix function A be defined on the complex plane by*

$$A(z) = \begin{pmatrix} 4z^2 & 1 & -1 \\ -1 & 2z^2 & 0 \\ 3 & 0 & 1 \end{pmatrix}.$$

How many distinct values of z are there such that $|z| < 1$ and $A(z)$ is not invertible?

Problem 9 *Let $\mathbb{Z}^2$ be the group of lattice points in the plane (ordered pairs of integers, with coordinatewise addition as the group operation). Let H_1 be the subgroup generated by the two elements $(1,2)$ and $(4,1)$, and H_2 the subgroup generated by the two elements $(3,2)$ and $(1,3)$. Are the quotient groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ isomorphic?*

Problem 10 *Suppose addition and multiplication are defined on $\mathbb{C}^n$, complex n-space, coordinatewise, making $\mathbb{C}^n$ into a ring. Find all ring homomorphisms of $\mathbb{C}^n$ onto $\mathbb{C}$.*

Problem 11 *Let the complex valued functions f_n, $n \in \mathbb{Z}$, be defined on $\mathbb{R}$ by*

$$f_n(x) = \frac{(x-i)^n}{\sqrt{\pi}(x+i)^{n+1}}\,.$$

Prove that these functions are orthonormal*; that is,*

$$\int_{-\infty}^{\infty} f_m(x)\overline{f_n(x)}\, dx = \begin{cases} 1 & \text{if} \quad m = n \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

Problem 12 *Let f be a real valued continuous function on $\mathbb{R}$ satisfying the* mean value inequality *below:*

$$f(x) \leqslant \frac{1}{2h} \int_{x-h}^{x+h} f(y)\,dy, \qquad x \in \mathbb{R}, \qquad h > 0.$$

Prove:

1. *The maximum of f on any closed interval is assumed at one of the endpoints.*

2. *f is convex.*

Problem 13 *Let S be a nonempty commuting set of $n \times n$ complex matrices ($n \geqslant 1$). Prove that the members of S have a common eigenvector.*

Problem 14 *Let K be a compact subset of $\mathbb{R}^n$ and $\{B_j\}$ a sequence of open balls that covers K. Prove that there is a positive number ε such that each ε-ball centered at a point of K is contained in one of the balls B_j.*

Problem 15 *Consider $\mathbb{R}^2$ be equipped with the Euclidean metric$d(x, y) = \|x - y\|$. Let T be an isometry of $\mathbb{R}^2$ into itself. Prove that T can be represented as $T(x) = a + U(x)$, where a is a vector in $\mathbb{R}^2$ and U is an orthogonal linear transformation.*

Problem 16 *Let $\mathbb{Z}$ be the ring of integers, p a prime, and$\mathbf{F}_p = \mathbb{Z}/p\mathbb{Z}$ the field of p elements. Let x be an indeterminate, and set$R_1 = \mathbf{F}_p[x]/\langle x^2 - 2\rangle$, $R_2 = \mathbf{F}_p[x]/\langle x^2 - 3\rangle$. Determine whether the rings R_1 and R_2 are isomorphic in each of the cases $p = 2, 5, 11$.*

Problem 17 *Let V be a finite-dimensional vector space (over $\mathbb{C}$) of C^∞ complex valued functions on $\mathbb{R}$ (the linear operations being defined pointwise). Prove that if V is closed under differentiation (i.e., $f'(x)$ belongs to V whenever $f(x)$ does), then V is closed under translations (i.e., $f(x + a)$ belongs to V whenever $f(x)$ does, for all real numbers a).*

Problem 18 *Let f, g_1, $g_2, \ldots$ be entire functions. Assume that*

1. *$|g_n^{(k)}(0)| \leqslant |f^{(k)}(0)|$ for all n and k;*

2. *$\lim_{n\to\infty} g_n^{(k)}(0)$ exists for all k.*

Prove that the sequence $\{g_n\}$ converges uniformly on compact sets and that its limit is an entire function.

Problem 19 *Prove that the additive group of $\mathbb{Q}$, the rational number field, is not finitely generated.*
Note: See also Problems **??** *and* **??***.*

Problem 20 *Evaluate*

$$\int_{|z|=1} (e^{2\pi z}+1)^{-2}\, dz$$

where the integral is taken in counterclockwise direction.