Preliminary Exam - Spring 1985

Problem 1 *Let $f(x)$, $0 \leqslant x < \infty$, be continuous, differentiable, with $f(0) = 0$, and that $f'(x)$ is an increasing function of x for $x \geqslant 0$. Prove that*

$$g(x) = \begin{cases} f(x)/x, & x > 0 \\ f'(0), & x = 0 \end{cases}$$

is an increasing function of x.

Problem 2 *In a commutative group G, let the element a have order r, let b have order s $(r, s < \infty)$, and assume that the greatest common divisor of r and s is 1. Show that ab has order rs.*

Problem 3 *Show that a necessary and sufficient condition for three points a, b, and c in the complex plane to form an equilateral triangle is that*

$$a^2 + b^2 + c^2 = bc + ca + ab.$$

Problem 4 *Let $R > 1$ and let f be analytic on $|z| < R$ except at $z = 1$, where f has a simple pole. If*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z| < 1)$$

is the Maclaurin series for f, show that $\lim_{n\to\infty} a_n$ exists.

Problem 5 *Factor $x^4 + x^3 + x + 3$ completely in $\mathbb{Z}_5[x]$.*

Problem 6 *Let A and B be two $n \times n$ self-adjoint (i.e., Hermitian) matrices over $\mathbb{C}$ such that all eigenvalues of A lie in $[a, a']$ and all eigenvalues of B lie in $[b, b']$. Show that all eigenvalues of $A + B$ lie in $[a + b, a' + b']$.*

Problem 7 *Prove that*

$$\int_0^\infty e^{-x^2} \cos(2bx)\, dx = \frac{1}{2}\sqrt{\pi}e^{-b^2}.$$

What restrictions, if any, need be placed on b?

Problem 8 *Let $h > 0$ be given. Consider the linear difference equation*

$$\frac{y\left((n+2)h\right) - 2y\left((n+1)h\right) + y(nh)}{h^2} = -y(nh), \quad n = 0, 1, 2, \ldots.$$

(Note the analogy with the differential equation $y'' = -y$.)

1. *Find the general solution of the equation by trying suitable exponential substitutions.*

2. *Find the solution with $y(0) = 0$ and $y(h) = h$. Denote it by $S_h(nh)$,$n = 1, 2, \ldots$.*

3. *Let x be fixed and $h = \dfrac{x}{n}$. Show that*

$$\lim_{n\to\infty} S_{\frac{x}{n}}\left(\frac{nx}{n}\right) = \sin x\,.$$

Problem 9 *Define the function ζ by*

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Prove that $\zeta(x)$ is defined and has continuous derivatives of all orders in the interval $1 < x < \infty$.

Problem 10 *For arbitrary elements a, b, and c in a field $\mathbf{F}$, compute the minimal polynomial of the matrix*

$$\begin{pmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{pmatrix}.$$

Problem 11 *Let $\mathbf{F} = \{a + b\sqrt[3]{2} + c\sqrt[3]{4} \mid a, b, c \in \mathbb{Q}\}$. Prove that $\mathbf{F}$ is a field and each element in $\mathbf{F}$ has a unique representation as $a + b\sqrt[3]{2} + c\sqrt[3]{4}$ with $a, b, c \in \mathbb{Q}$. Find $(1 - \sqrt[3]{2})^{-1}$ in $\mathbf{F}$.*

Problem 12 *Prove that*

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}\, dx = \frac{\pi}{\sin \pi\alpha}.$$

What restrictions must be placed on α?

Problem 13 *Prove that for any* $a \in \mathbb{C}$ *and any integer* $n \geqslant 2$*, the equation* $1 + z + az^n = 0$ *has at least one root in the disc* $|z| \leqslant 2$.

Problem 14 *Show that*

$$I = \int_0^\pi \log(\sin x)\, dx$$

converges as an improper Riemann integral. Evaluate I.

Problem 15 *Let* $\zeta = e^{\frac{2\pi i}{7}}$ *be a primitive* 7^{th} *root of unity. Find a cubic polynomial with integer coefficients having* $\alpha = \zeta + \zeta^{-1}$ *as a root.*

Problem 16 *Let* f *be continuous on* $\mathbb{R}$*, and let*

$$f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f\left(x + \frac{k}{n}\right).$$

Prove that $f_n(x)$ *converges uniformly to a limit on every finite interval* $[a, b]$.

Problem 17 *Let* v_1 *and* v_2 *be two real valued continuous functions on* $\mathbb{R}$ *such that* $v_1(x) < v_2(x)$ *for all* $x \in \mathbb{R}$*. Let* $\varphi_1(t)$ *and* $\varphi_2(t)$ *be, respectively, solutions of the differential equations*

$$\frac{dx}{dt} = v_1(x) \quad \textit{and} \quad \frac{dx}{dt} = v_2(x)$$

for $a < t < b$*. If* $\varphi_1(t_0) = \varphi_2(t_0)$ *for some* $t_0 \in (a, b)$*, show that* $\varphi_1(t) \leqslant \varphi_2(t)$ *for all* $t \in (t_0, b)$.

Problem 18 *Let* A *and* B *be two* $n \times n$ *self-adjoint (i.e., Hermitian) matrices over* $\mathbb{C}$ *and assume* A *is positive definite. Prove that all eigenvalues of* AB *are real.*

Problem 19 *Let* $\mathbf{F}$ *be a finite field. Give a complete proof of the fact that the number of elements of* $\mathbf{F}$ *is of the form* p^r*, where* $p \geqslant 2$ *is a prime number and* r *is an integer* $\geqslant 1$.

Problem 20 *Let* $f(z)$ *be an analytic function that maps the open disc* $|z| < 1$ *into itself. Show that* $|f'(z)| \leqslant 1/(1 - |z|^2)$.