Preliminary Exam - Spring 1984

Problem 1 *Evaluate*

$$\int_0^\infty \frac{\log x}{a^2 + x^2}\, dx$$

for $a > 0$.

Problem 2 *For a p-group of order* p^4, *assume the center of* G *has order* p^2. *Determine the number of conjugacy classes of* G.

Problem 3 *Let* $f : [0,1] \to \mathbb{R}$ *be continuous function, with*$f(0) = f(1) = 0$. *Assume that* f'' *exists on* $0 < x < 1$, *with* $f'' + 2f' + f \geqslant 0$. *Show that* $f(x) \leqslant 0$ *for all* $0 \leqslant x \leqslant 1$.

Problem 4 *Which number is larger,* π^3 *or* 3^π*?*

Problem 5 *Let* A *and* B *be complex* $n \times n$ *matrices such that*$AB = BA^2$, *and assume* A *has no eigenvalues of absolute value* 1. *Prove that* A *and* B *have a common (nonzero) eigenvector.*

Problem 6 *Let* a *be a positive real number. Define a sequence* (x_n) *by*

$$x_0 = 0, \quad x_{n+1} = a + x_n^2, \quad n \geqslant 0\,.$$

Find a necessary and sufficient condition on a *in order that a finite limit* $\lim\limits_{n\to\infty} x_n$ *should exist.*

Problem 7 *Find the number of roots of*

$$z^7 - 4z^3 - 11 = 0$$

which lie between the two circles $|z| = 1$ *and* $|z| = 2$.

Problem 8 *Show that the system of differential equations*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

has a solution which tends to ∞ *as* $t \to -\infty$ *and tends to the origin as* $t \to +\infty$.

Problem 9 *Let A be a real $m \times n$ matrix with rational entries and let b be an m-tuple of rational numbers. Assume that the system of equations $Ax = b$ has a solution x in complex n-space $\mathbb{C}^n$. Show that the equation has a solution vector with rational components, or give a counterexample.*

Problem 10 *Let R be a principal ideal domain and let $\mathfrak{I}$ and $\mathfrak{J}$ be nonzero ideals in R. Show that $\mathfrak{I}\mathfrak{J} = \mathfrak{I} \cap \mathfrak{J}$ if and only if $\mathfrak{I} + \mathfrak{J} = R$.*

Problem 11 *Prove the following statement or supply a counterexample: If A and B are real $n \times n$ matrices which are similar over $\mathbb{C}$, then A and B are similar over $\mathbb{R}$.*

Problem 12 *Consider the equation*

$$\frac{dy}{dx} = y - \sin y.$$

Show that there is an $\varepsilon > 0$ such that if $|y_0| < \varepsilon$, then the solution $y = f(x)$ with $f(0) = y_0$ satisfies

$$\lim_{x \to -\infty} f(x) = 0.$$

Problem 13 *Let I be an open interval in $\mathbb{R}$ containing zero. Assume that f' exists on a neighborhood of zero and $f''(0)$ exists. Show that*

$$f(x) = f(0) + f'(0)\sin x + \frac{1}{2}f''(0)\sin^2 x + o(x^2)$$

($o(x^2)$ denotes a quantity such that $\dfrac{o(x^2)}{x^2} \to 0$ as $x \to 0$).

Problem 14 *Let $\mathbf{F}$ be a field and let X be a finite set. Let $R(X, \mathbf{F})$ be the ring of all functions from X to $\mathbf{F}$, endowed with the pointwise operations. What are the maximal ideals of $R(X, \mathbf{F})$?*

Problem 15 *Let F be a continuous complex valued function on the interval $[0, 1]$. Let*

$$f(z) = \int_0^1 \frac{F(t)}{t - z}\, dt,$$

for z a complex number not in $[0, 1]$.

1. *Prove that f is an analytic function.*

2. *Express the coefficients of the Laurent series of f about ∞ in terms of F. Use the result to show that F is uniquely determined by f.*

Problem 16 *Prove, or supply a counterexample: If A is an invertible $n \times n$ complex matrix and some power of A is diagonal, then A can be diagonalized.*

Problem 17 *Prove that the Taylor coefficients at the origin of the function*

$$f(z) = \frac{z}{e^z - 1}$$

are rational numbers.

Problem 18 *Prove or supply a counterexample: If the function f from $\mathbb{R}$ to $\mathbb{R}$ has both a left limit and a right limit at each point of $\mathbb{R}$, then the set of discontinuities of f is, at most, countable.*

Problem 19 *Let $f(x) = x \log(1 + x^{-1})$, $0 < x < \infty$.*

1. *Show that f is strictly monotonically increasing.*

2. *Compute $\lim f(x)$ as $x \to 0$ and $x \to \infty$.*

Problem 20 *Determine all finitely generated abelian groups G which have only finitely many automorphisms.*