Preliminary Exam - Spring 1983

Problem 1 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone decreasing function, defined on the positive real numbers with*

$$\int_0^\infty f(x)\,dx < \infty.$$

Show that

$$\lim_{x\to\infty} xf(x) = 0.$$

Problem 2 *Let $A = (a_{ij})$ be an $n\times n$ real matrix satisfying the conditions:*

$$a_{ii} > 0 \quad (1 \leqslant i \leqslant n),$$

$$a_{ij} \leqslant 0 \quad (i \neq j,\ 1 \leqslant i, j \leqslant n),$$

$$\sum_{i=1}^{n} a_{ij} > 0 \quad (1 \leqslant j \leqslant n).$$

Show that $\det(A) > 0$.

Problem 3 *A fractional linear transformation maps the annulus$r < |z| < 1$ (where $r > 0$) onto the domain bounded by the two circles $|z - \frac{1}{4}| = \frac{1}{4}$ and $|z| = 1$. Find r.*

Problem 4 *In the triangular network in $\mathbb{R}^2$ depicted below, the points P_0, P_1, P_2, and P_3 are respectively $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. Describe the structure of the group of all Euclidean transformations of $\mathbb{R}^2$ which leave this network invariant.*

`../Fig/Pr/Sp83-4.ps` *not found*

Problem 5 *Find all solutions $y : \mathbb{R} \to \mathbb{R}$ to*

$$\frac{dy}{dx} = \sqrt{y\,(y-2)}, \qquad y(0) = 0.$$

Problem 6 *Suppose that f is a continuous function on $\mathbb{R}$ which is periodic with period 1, i.e., $f(x+1) = f(x)$. Show:*

1. *The function f is bounded above and below and achieves its maximum and minimum.*
2. *The function f is uniformly continuous on $\mathbb{R}$.*
3. *There exists a real number x_0 such that*
$$f(x_0 + \pi) = f(x_0).$$

Problem 7 *Let H be the group of integers $\bmod p$, under addition, where p is a prime number. Suppose that n is an integer satisfying $1 \leqslant n \leqslant p$, and let G be the group $H \times H \times \cdots \times H$ (n factors). Show that G has no automorphism of order p^2.*

Problem 8 *Suppose that $n > 1$ is an integer. Prove that the sum*

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

is not an integer.

Problem 9 *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies*

$$\|F(x) - F(y)\| \geqslant \lambda \|x - y\|$$

for all $x, y \in \mathbb{R}^n$ and some $\lambda > 0$. Prove that F is one-to-one, onto, and has a continuous inverse.
Note: See also Problem **??**.

Problem 10 *Evaluate*

$$\int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx$$

Problem 11 *Let M be an invertible real $n \times n$ matrix. Show that there is a decomposition $M = UT$ in which U is an $n \times n$ real orthogonal matrix and T is upper-triangular with positive diagonal entries. Is this decomposition unique?*

Problem 12 *Determine all the complex analytic functions f defined on the unit disc $\mathbb{D}$ which satisfy*

$$f''\left(\frac{1}{n}\right) + f\left(\frac{1}{n}\right) = 0$$

for $n = 2, 3, 4, \ldots$.

Problem 13 *Let $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ be real numbers. Show that the infinite series*

$$\sum_{n=1}^{\infty} \frac{e^{i\lambda_n x}}{n^2}$$

converges uniformly over $\mathbb{R}$ to a continuous limit function $f : \mathbb{R} \to \mathbb{C}$. Show, further, that the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx$$

exists.

Problem 14 *Let G be an abelian group which is generated by, at most, n elements. Show that each subgroup of G is again generated by, at most, n elements.*

Problem 15 *Let f and g be complex polynomials with the degree of g at least two more than the degree of f. Show that there is a positive number r such that*

$$\int_C \frac{f(z)}{g(z)}\, dz = 0$$

for each simple closed curve C which does not intersect $\{z \mid |z| \leqslant r\}$.

Problem 16 *Let V be a real vector space of dimension n, and let$S : V \times V \to \mathbb{R}$ be a nondegenerate bilinear form. Suppose that W is a linear subspace of V such that the restriction of S to $W \times W$ is identically 0. Show that $\dim W \leqslant n/2$.*

Problem 17 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x-\pi)}\, dx\,.$$

Problem 18 *Let* $\mathbf{F}$ *be the field with seven elements. How many 3×3 matrices with coefficients in* $\mathbf{F}$ *have determinant* 2*? How many have determinant* 3*?*

Problem 19 *Show that the initial value problem*

$$x' = 1 + 5\cos x, \qquad x(0) = 7$$

has a solution defined on all of $\mathbb{R}$.

Problem 20 *Show that the interval* $[0, 1]$ *cannot be written as a countably infinite disjoint union of closed subintervals of* $[0, 1]$.