Preliminary Exam - Spring 1982

Problem 1 *Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial with complex coefficients has a complex root.*

Problem 2 *Let $S \subset \mathbb{R}^n$ be a subset which is uncountable. Prove that there is a sequence of distinct points in S converging to a point of S.*

Problem 3 *Let A and B be $n \times n$ complex matrices. Prove that*

$$|\operatorname{tr}(AB^*)|^2 \leqslant \operatorname{tr}(AA^*)\operatorname{tr}(BB^*).$$

Problem 4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ have directional derivatives in all directions at the origin. Is f differentiable at the origin? Prove or give a counterexample.*

Problem 5 *Let $\{g_n\}$ be a sequence of twice differentiable functions on $[0, 1]$ such that $g_n(0) = g_n'(0) = 0$ for all n. Suppose also that$|g_n''(x)| \leqslant 1$ for all n and all $x \in [0, 1]$. Prove that there is a subsequence of $\{g_n\}$ which converges uniformly on $[0, 1]$.*

Problem 6 *Suppose that $f(x)$ is a polynomial with real coefficients and a is a real number with $f(a) \neq 0$. Show that there exists a real polynomial $g(x)$ such that if we define p by $p(x) = f(x)g(x)$, we have $p(a) = 1$, $p'(a) = 0$, and $p''(a) = 0$.*

Problem 7 *Suppose that the group G is generated by elements x and y that satisfy $x^5y^3 = x^8y^5 = 1$. Is G the trivial group?*

Problem 8 *Find*

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1}\, dx$$

by contour integration.

Problem 9 *Find the Jordan Canonical Form for the matrix (over $\mathbb{R}$)*

$$\begin{pmatrix} 4 & 1 & 0 \\ -4 & 0 & 0 \\ 19 & 17 & 5 \end{pmatrix}.$$

Problem 10 *Prove that any group of order* 77 *is cyclic.*

Problem 11 *Decide, without too much computation, whether a finite limit*

$$\lim_{z\to 0}\left((\tan z)^{-2} - z^{-2}\right)$$

exists, where z is a complex variable, and if yes, compute the limit.

Problem 12 *Prove or give a counterexample: Every connected, locally pathwise connected set in $\mathbb{R}^n$ is pathwise connected.*

Problem 13 *Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces. Prove that*

$$\dim(\ker T) + \dim(\operatorname{range} T) = \dim V\,.$$

Problem 14 *Let $f : I \to \mathbb{R}$ (where I is an interval of $\mathbb{R}$) be such that $f(x) > 0,\ x \in I$. Suppose that $e^{cx} f(x)$ is convex in I for every real number c. Show that $\log f(x)$ is convex in I.*
Note: A function $g : I \to \mathbb{R}$ is convex *if*

$$g\left(tx + (1-t)y\right) \leqslant tg(x) + (1-t)g(y)$$

for all x and y in I and $0 \leqslant t \leqslant 1$.

Problem 15 *How many nonsingular 2×2 matrices are there over the field of p elements?*

Problem 16 *Prove that if G is a group containing no subgroup of index* 2*, then any subgroup of index* 3 *is normal.*

Problem 17 *Let $\{f_n\}$ be a sequence of continuous functions from $[0,1]$ to $\mathbb{R}$. Suppose that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in [0,1]$ and also that, for some constant K, we have*

$$\left|\int_0^1 f_n(x)\,dx\right| \leqslant K < \infty$$

for all n. Does

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 0\,?$$

Problem 18 *For* $\Re z \geqslant 0$*, define*

$$F(z) = \int_0^\infty \frac{e^{-zt}}{1+t^4}\,dt.$$

Show that F(z) is continuous for $\Re z \geqslant 0$ *and analytic for* $\Re z > 0$*.*

Problem 19 *Show that the initial value problem*

$$y'(x) = 2 + 3\sin(y(x)), \quad y(0) = 4$$

has a solution defined for $-\infty < x < \infty$*.*

Problem 20 *Prove that the polynomial* $x^4 + x + 1$ *is irreducible over* $\mathbb{Q}$*.*