Preliminary Exam - Spring 1981

Problem 1 *Let $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ be the usual unit vectors in $\mathbb{R}^3$. Let $\vec{F}$ denote the vector field*

$$(x^2 + y - 4)\vec{\imath} + 3xy\vec{\jmath} + (2xz + z^2)\vec{k}.$$

1. *Compute $\nabla \times \vec{F}$ (the curl of $\vec{F}$).*

2. *Compute the integral of $\nabla \times \vec{F}$ over the surface $x^2 + y^2 + z^2 = 16$, $z \geqslant 0$.*

Problem 2 *Let T be a linear transformation of a vector space V into itself. Suppose $x \in V$ is such that $T^m x = 0$, $T^{m-1}x \neq 0$ for some positive integer m. Show that $x, Tx, \ldots, T^{m-1}x$ are linearly independent.*

Problem 3 *Let D be an ordered integral domain and $a \in D$. Prove that*

$$a^2 - a + 1 > 0.$$

Problem 4 *Consider the system of differential equations*

$$\begin{aligned} \frac{dx}{dt} &= y + x(1 - x^2 - y^2) \\ \frac{dy}{dt} &= -x + y(1 - x^2 - y^2). \end{aligned}$$

1. *Show that for any x_0 and y_0, there is a unique solution $(x(t), y(t))$ defined for all $t \in \mathbb{R}$ such that $x(0) = x_0$, $y(0) = y_0$.*

2. *Show that if $x_0 \neq 0$ and $y_0 \neq 0$, the solution referred to in Part 1 approaches the circle $x^2 + y^2 = 1$ as $t \to \infty$.*

Problem 5 *Decompose $x^4 - 4$ and $x^3 - 2$ into irreducibles over $\mathbb{R}$, over $\mathbb{Z}$, and over $\mathbb{Z}_3$ (the integers modulo 3).*

Problem 6 *Suppose the complex polynomial*

$$\sum_{k=0}^{n} a_k z^k$$

has n distinct roots $r_1, \ldots, r_n \in \mathbb{C}$. Prove that if $|b_k - a_k|$ is sufficiently small then

$$\sum_{k=0}^{n} b_k z^k$$

has n roots which are smooth functions of $b_0, \ldots, b_n$.

Problem 7 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(1+x^2)^2}\, dx\,.$$

Problem 8 *Let $f : [0,1] \to \mathbb{R}$ be continuous and $k \in \mathbb{N}$. Prove that there is a real polynomial $P(x)$ of degree $\leqslant k$ which minimizes (for all such polynomials)*

$$\sup_{0 \leqslant x \leqslant 1} |f(x) - P(x)|.$$

Problem 9 *Show that the following three conditions are all equivalent for a real 3×3 symmetric matrix A, whose eigenvalues are λ_1, λ_2, and λ_3:*

1. $\operatorname{tr} A$ *is not an eigenvalue of* A.
2. $(a+b)(b+c)(a+c) \neq 0$.
3. *The map $L : S \to S$ is an isomorphism, where S is the space of 3×3 real skew-symmetric matrices and $L(W) = AW + WA$.*

Problem 10

1. *Give an example of a sequence of C^1 functions*

$$f_k : [0,\infty) \to \mathbb{R}, \quad k = 0, 1, 2, \ldots$$

such that $f_k(0) = 0$ for all k, and $f_k'(x) \to f_0'(x)$ for all x as $k \to \infty$, but $f_k(x)$ does not converge to $f_0(x)$ for all x as $k \to \infty$.

2. *State an extra condition which would imply that $f_k(x) \to f_0(x)$ for all x as $k \to \infty$.*

Problem 11 *Evaluate*

$$\int_C \frac{e^z - 1}{z^2(z-1)}\, dz$$

where C is the closed curve shown below:

../Fig/Pr/Sp81-11.ps *not found*

Problem 12 *For $x \in \mathbb{R}$, let*

$$A_x = \begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}.$$

1. *Prove that* $\det(A_x) = (x-1)^3(x+3)$.
2. *Prove that if $x \neq 1, -3$, then $A_x^{-1} = -(x-1)^{-1}(x+3)^{-1}A_{-x-2}$.*

Problem 13 *Which of the following series converges?*

1.
$$\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}.$$

2.
$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}.$$

Problem 14 *The set of real 3×3 symmetric matrices is a real, finite-dimensional vector space isomorphic to $\mathbb{R}^6$. Show that the subset of such matrices of signature $(2,1)$ is an open connected subspace in the usual topology on $\mathbb{R}^6$.*

Problem 15 *Let $\mathbf{M}$ be one of the following fields: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbf{F}_9$ (the field with nine elements). Let $\mathfrak{I} \subset \mathbf{M}[x]$ be the ideal generated by $x^4 + 2x - 2$. For which choices of $\mathbf{M}$ is the ring $\mathbf{M}[x]/\mathfrak{I}$ a field?*

Problem 16 *Let $f(x)$ be a real valued function defined for all $x \geqslant 1$, satisfying $f(1) = 1$ and*

$$f'(x) = \frac{1}{x^2 + f(x)^2}.$$

Prove that

$$\lim_{x\to\infty} f(x)$$

exists and is less than $1 + \frac{\pi}{4}$.

Problem 17 *Let b be a real nonzero $n \times 1$ matrix (a column vector). Set $M = bb^t$ (an $n \times n$ matrix) where b^t denotes the transpose of b.*

1. *Prove that there is an orthogonal matrix Q such that $QMQ^{-1} = D$ is diagonal, and find D.*

2. *Describe geometrically the linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$.*

Problem 18 *Describe the two regions in (a, b)-space for which the function*

$$f_{a,b}(x, y) = ay^2 + bx$$

restricted to the circle $x^2 + y^2 = 1$, has exactly two, and exactly four critical points, respectively.

Problem 19 *Let G be a finite group. A* conjugacy class *is a set of the form*

$$C(a) = \{bab^{-1} \mid b \in G\}$$

for some $a \in G$.

1. *Prove that the number of elements in a conjugacy class divides the order of G.*

2. *Do all conjugacy classes have the same number of elements?*

3. *If G has only two conjugacy classes, prove G has order 2.*

Problem 20 *Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = 0$. Show there is a continuous concave function $g : [0, 1] \to \mathbb{R}$ such that $g(0) = 0$ and $g(x) \geqslant f(x)$ for all $x \in [0, 1]$.*
Note: A function $g : I \to \mathbb{R}$ is concave *if*

$$g\left(tx + (1 - t)y\right) \geqslant tg(x) + (1 - t)g(y)$$

for all x and y in I and $0 \leqslant t \leqslant 1$.