Problem 1 Let \( \vec{i}, \vec{j}, \) and \( \vec{k} \) be the usual unit vectors in \( \mathbb{R}^3 \). Let \( \vec{F} \) denote the vector field 
\[
(x^2 + y - 4)\vec{i} + 3xy\vec{j} + (2xz + z^2)\vec{k}.
\]
1. Compute \( \nabla \times \vec{F} \) (the curl of \( \vec{F} \)).
2. Compute the integral of \( \nabla \times \vec{F} \) over the surface \( x^2 + y^2 + z^2 = 16, \ z \geq 0 \).

Problem 2 Let \( T \) be a linear transformation of a vector space \( V \) into itself. Suppose \( x \in V \) is such that \( T^m x = 0, T^{m-1} x \neq 0 \) for some positive integer \( m \). Show that \( x, Tx, \ldots, T^{m-1} x \) are linearly independent.

Problem 3 Let \( D \) be an ordered integral domain and \( a \in D \). Prove that
\[
a^2 - a + 1 > 0.
\]

Problem 4 Consider the system of differential equations 
\[
\begin{align*}
\frac{dx}{dt} &= y + x(1 - x^2 - y^2) \\
\frac{dy}{dt} &= -x + y(1 - x^2 - y^2).
\end{align*}
\]
1. Show that for any \( x_0 \) and \( y_0 \), there is a unique solution \( (x(t), y(t)) \) defined for all \( t \in \mathbb{R} \) such that \( x(0) = x_0, y(0) = y_0 \).
2. Show that if \( x_0 \neq 0 \) and \( y_0 \neq 0 \), the solution referred to in Part 1 approaches the circle \( x^2 + y^2 = 1 \) as \( t \to \infty \).

Problem 5 Decompose \( x^4 - 4 \) and \( x^3 - 2 \) into irreducibles over \( \mathbb{R} \), over \( \mathbb{Z} \), and over \( \mathbb{Z}_3 \) (the integers modulo 3).

Problem 6 Suppose the complex polynomial
\[
\sum_{k=0}^{n} a_k z^k
\]
has \( n \) distinct roots \( r_1, \ldots, r_n \in \mathbb{C} \). Prove that if \( |b_k - a_k| \) is sufficiently small then
\[
\sum_{k=0}^{n} b_k z^k
\]
has \( n \) roots which are smooth functions of \( b_0, \ldots, b_n \).

**Problem 7** Evaluate
\[
\int_{-\infty}^{\infty} \frac{x \sin x}{(1 + x^2)^2} \, dx.
\]

**Problem 8** Let \( f : [0, 1] \to \mathbb{R} \) be continuous and \( k \in \mathbb{N} \). Prove that there is a real polynomial \( P(x) \) of degree \( \leq k \) which minimizes (for all such polynomials)
\[
\sup_{0 \leq x \leq 1} |f(x) - P(x)|.
\]

**Problem 9** Show that the following three conditions are all equivalent for a real \( 3 \times 3 \) symmetric matrix \( A \), whose eigenvalues are \( \lambda_1, \lambda_2, \) and \( \lambda_3 \):

1. \( \text{tr} A \) is not an eigenvalue of \( A \).
2. \( (a + b)(b + c)(a + c) \neq 0 \).
3. The map \( L : S \to S \) is an isomorphism, where \( S \) is the space of \( 3 \times 3 \) real skew-symmetric matrices and \( L(W) = AW + WA \).

**Problem 10**
1. Give an example of a sequence of \( C^1 \) functions
\[
f_k : [0, \infty) \to \mathbb{R}, \quad k = 0, 1, 2, \ldots
\]
such that \( f_k(0) = 0 \) for all \( k \), and \( f'_k(x) \to f'_0(x) \) for all \( x \) as \( k \to \infty \), but \( f_k(x) \) does not converge to \( f_0(x) \) for all \( x \) as \( k \to \infty \).

2. State an extra condition which would imply that \( f_k(x) \to f_0(x) \) for all \( x \) as \( k \to \infty \).

**Problem 11** Evaluate
\[
\int_C \frac{e^z - 1}{z^2(z - 1)} \, dz
\]
where \( C \) is the closed curve shown below:

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Problem 12  For $x \in \mathbb{R}$, let

\[
A_x = \begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}.
\]

1. Prove that $\det(A_x) = (x - 1)^3(x + 3)$.
2. Prove that if $x \neq 1, -3$, then $A_x^{-1} = -(x - 1)^{-1}(x + 3)^{-1}A_{x-2}$.

Problem 13  Which of the following series converges?

1. $\sum_{n=1}^{\infty} \frac{(2n)!(3n)!}{n!(4n)!}$.

2. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$.

Problem 14  The set of real $3 \times 3$ symmetric matrices is a real, finite-dimensional vector space isomorphic to $\mathbb{R}^6$. Show that the subset of such matrices of signature $(2, 1)$ is an open connected subspace in the usual topology on $\mathbb{R}^6$.

Problem 15  Let $M$ be one of the following fields: $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, and $\mathbb{F}_9$ (the field with nine elements). Let $J \subset M[x]$ be the ideal generated by $x^4 + 2x - 2$. For which choices of $M$ is the ring $M[x]/J$ a field?

Problem 16  Let $f(x)$ be a real valued function defined for all $x \geq 1$, satisfying $f(1) = 1$ and

\[
f'(x) = \frac{1}{x^2 + f(x)^2}.
\]

Prove that

\[
\lim_{x \to \infty} f(x)
\]

exists and is less than $1 + \frac{\pi}{4}$. 

Problem 17 Let $b$ be a real nonzero $n \times 1$ matrix (a column vector). Set $M = bb^t$ (an $n \times n$ matrix) where $b^t$ denotes the transpose of $b$.

1. Prove that there is an orthogonal matrix $Q$ such that $QMQ^{-1} = D$ is diagonal, and find $D$.

2. Describe geometrically the linear transformation $M : \mathbb{R}^n \to \mathbb{R}^n$.

Problem 18 Describe the two regions in $(a, b)$-space for which the function $f_{a,b}(x, y) = ay^2 + bx$ restricted to the circle $x^2 + y^2 = 1$, has exactly two, and exactly four critical points, respectively.

Problem 19 Let $G$ be a finite group. A conjugacy class is a set of the form $C(a) = \{bab^{-1} \mid b \in G\}$ for some $a \in G$.

1. Prove that the number of elements in a conjugacy class divides the order of $G$.

2. Do all conjugacy classes have the same number of elements?

3. If $G$ has only two conjugacy classes, prove $G$ has order 2.

Problem 20 Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = 0$. Show there is a continuous concave function $g : [0, 1] \to \mathbb{R}$ such that $g(0) = 0$ and $g(x) \geq f(x)$ for all $x \in [0, 1]$.

Note: A function $g : I \to \mathbb{R}$ is concave if

$$g(tx + (1 - t)y) \geq tg(x) + (1 - t)g(y)$$

for all $x$ and $y$ in $I$ and $0 \leq t \leq 1$. 