Preliminary Exam - Spring 1980

Problem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be the unique function such that$f(x) = x$ if $-\pi \leqslant x < \pi$ and $f(x + 2n\pi) = f(x)$ for all $n \in \mathbb{Z}$.*

1. *Prove that the Fourier series of f is*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2 \sin nx}{n}.$$

2. *Prove that the series does not converge uniformly.*

3. *For each $x \in \mathbb{R}$, find the sum of the series.*

Problem 2 *Let $f_n : \mathbb{R} \to \mathbb{R}$ be differentiable for each $n = 1, 2, \ldots$ with $|f_n'(x)| \leqslant 1$ for all n, x. Assume*

$$\lim_{n \to \infty} f_n(x) = g(x)$$

for all x. Prove that $g : \mathbb{R} \to \mathbb{R}$ is continuous.

Problem 3 *Let P_2 denote the set of real polynomials of degree $\leqslant 2$. Define the map $J : P_2 \to \mathbb{R}$ by*

$$J(f) = \int_0^1 f(x)^2 \, dx \, .$$

Let $Q = \{f \in P_2 \mid f(1) = 1\}$. Show that J attains a minimum value on Q and determine where the minimum occurs.

Problem 4 *Let $a > 0$ be a constant $\neq 2$. Let C_a denote the positively oriented circle of radius a centered at the origin. Evaluate*

$$\int_{C_a} \frac{z^2 + e^z}{z^2(z-2)} \, dz \, .$$

Problem 5 *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be an analytic function in the open unit disc $\mathbb{D}$. *Assume that*

$$\sum_{n=2}^{\infty} n|a_n| \leqslant |a_1| \quad \textit{with} \quad a_1 \neq 0.$$

Prove that f *is injective.*

Problem 6 G *is a group of order* n, H *a proper subgroup of order* m, *and* $(n/m)! < 2n$. *Prove* G *has a proper normal subgroup different from the identity.*

Problem 7 *Let* $n \geqslant 2$ *be an integer such that* $2^n + n^2$ *is prime. Prove that*

$$n \equiv 3 \pmod 6.$$

Problem 8 *Let* A *and* B *be* $n \times n$ *complex matrices. Prove or disprove each of the following statements:*

1. *If* A *and* B *are diagonalizable, so is* $A + B$.
2. *If* A *and* B *are diagonalizable, so is* AB.
3. *If* $A^2 = A$, *then* A *is diagonalizable.*
4. *If* A *is invertible and* A^2 *is diagonalizable, then* A *is diagonalizable.*

Problem 9 *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Show that every real matrix B *such that* $AB = BA$ *has the form* $sI + tA$, *where* $s, t \in \mathbb{R}$.

Problem 10 *Consider the differential equation*

$$x' = \frac{x^3 - x}{1 + e^x}.$$

1. *Find all its constant solutions.*

2. *Discuss* $\lim_{t\to\infty} x(t)$, *where* $x(t)$ *is the solution such that* $x(0) = \frac{1}{2}\cdot$

Problem 11 *Let* $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ *denote the unit sphere in* $\mathbb{R}^3$. *Evaluate the surface integral over* $\mathcal{S}$:

$$\iint_{\mathcal{S}} (x^2 + y + z)\, dA\,.$$

Problem 12 *Let* $M_{3\times 3}$ *denote the vector space of real 3×3 matrices. For any matrix* $A \in M_{3\times 3}$, *define the linear operator* $L_A : M_{3\times 3} \to M_{3\times 3}$, $L_A(B) = AB$. *Suppose that the determinant of* A *is* 32 *and the minimal polynomial is* $(t-4)(t-2)$. *What is the trace of* L_A?

Problem 13 *Let* G *be a subgroup of* S_n *the group of permutations of* n *objects. Assume* G *is* transitive; *that is, for any* x *and* y *in* S, *there is some* $\sigma \in G$ *with* $\sigma(x) = y$.

1. *Prove that* n *divides the order of* G.
2. *Suppose* $n = 4$. *For which integers* $k \geqslant 1$ *can such a* G *have order* $4k$?

Problem 14 *Find a real matrix* B *such that*

$$B^4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Problem 15 *Show that a vector space over an infinite field cannot be the union of a finite number of proper subspaces.*

Problem 16 *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be continuously differentiable. Assume the Jacobian matrix* $(\partial f_i / \partial x_j)$ *has rank* n *everywhere. Suppose* f *is* proper; *that is,* $f^{-1}(K)$ *is compact whenever* K *is compact. Prove* $f(\mathbb{R}^n) = \mathbb{R}^n$.

Problem 17 S_9 *is the group of permutations of* 9 *objects.*

1. *Exhibit an element of* S_9 *of order* 20.
2. *Prove that no element of* S_9 *has order* 18.

Problem 18 *For each $t \in \mathbb{R}$, let $P(t)$ be a symmetric real $n \times n$ matrix whose entries are continuous functions of t. Suppose for all t that the eigenvalues of $P(t)$ are all $\leqslant -1$. Let $x(t) = (x_1(t), \ldots, x_n(t))$ be a solution of the vector differential equation*

$$\frac{dx}{dt} = P(t)x.$$

Prove that

$$\lim_{t\to\infty} x(t) = 0.$$

Problem 19 *Let*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

be analytic in the disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. Assume f maps $\mathbb{D}$ one-to-one onto a domain G having area A. Prove

$$A = \pi \sum_{n=1}^{\infty} n|c_n|^2.$$

Problem 20 *Does there exist an analytic function mapping the annulus*

$$A = \{z \mid 1 \leqslant |z| \leqslant 4\}$$

onto the annulus

$$B = \{z \mid 1 \leqslant |z| \leqslant 2\}$$

and taking $C_1 \to C_1, C_4 \to C_2$, where C_r is the circle of radius r?