Preliminary Exam - Spring 1979

Problem 1 *Let $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be differentiable. Suppose*

$$\lim_{x \to 0} \frac{\partial f}{\partial x_j}(x)$$

exists for each $j = 1, \ldots, n$.

1. *Can f be extended to a continuous map from $\mathbb{R}^n$ to $\mathbb{R}$?*
2. *Assuming continuity at the origin, is f differentiable from $\mathbb{R}^n$ to $\mathbb{R}$?*

Problem 2 *Let E denote a finite-dimensional complex vector space with a Hermitian inner product $\langle x, y \rangle$.*

1. *Prove that E has an orthonormal basis.*
2. *Let $f : E \to \mathbb{C}$ be such that $f(x, y)$ is linear in x and conjugate linear in y. Show there is a linear map $A : E \to E$ such that $f(x, y) = \langle Ax, y \rangle$.*

Problem 3 *Let S_7 be the group of permutations of a set of seven objects. Find all n such that some element of S_7 has order n.*

Problem 4 *Prove that every compact metric space has a countable dense subset.*

Problem 5 *Find all solutions to the differential equation*

$$\frac{dy}{dx} = \sqrt{y}, \qquad y(0) = 0\,.$$

Problem 6 *Prove that if $1 < \lambda < \infty$, the function*

$$f_\lambda(z) = z + \lambda - e^z$$

has only one zero in the half-plane $\Re z < 0$, and that this zero is real.

Problem 7 *Evaluate*

$$\int_0^\infty \frac{x^2+1}{x^4+1}\,dx\,.$$

Problem 8 *Let M be a real nonsingular 3×3 matrix. Prove there are real matrices S and U such that $M = SU = US$, all the eigenvalues of U equal 1, and S is diagonalizable over $\mathbb{C}$.*

Problem 9 *Let M be an $n \times n$ complex matrix. Let G_M be the set of complex numbers λ such that the matrix λM is similar to M.*

1. *What is G_M if*

$$M = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,?$$

2. *Assume M is* not *nilpotent. Prove G_M is finite.*

Problem 10 *Let $f(x)$ be a polynomial over $\mathbb{Z}_p$, the field of integers* mod$p$. *Let $g(x) = x^p - x$. Show that the greatest common divisor of $f(x)$ and $g(x)$ is the product of the distinct linear factors of $f(x)$.*

Problem 11 *Classify all abelian groups of order* 80 *up to isomorphism.*

Problem 12 *Let G be a group with three normal subgroups N_1, N_2, and N_3. Suppose $N_i \cap N_j = \{e\}$ and $N_iN_j = G$ for all i, j with $i \neq j$. Show that G is abelian and N_i is isomorphic to N_j for all i, j.*

Problem 13 *Consider the system of differential equations:*

$$\begin{aligned} \frac{dx}{dt} &= y + tz \\ \frac{dy}{dt} &= z + t^2x \\ \frac{dz}{dt} &= x + e^t y. \end{aligned}$$

Prove there exists a solution defined for all $t \in [0,1]$, such that

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and also

$$\int_0^1 \left(x(t)^2 + y(t)^2 + z(t)^2\right) dt = 1.$$

Problem 14 *Let $M_{n\times n}$ denote the vector space of $n \times n$ real matrices for $n \geqslant 2$. Let $\det : M_{n\times n} \to \mathbb{R}$ be the determinant map.*

1. *Show that* $\det$ *is* C^∞.
2. *Show that the derivative of* $\det$ *at $A \in M_{n\times n}$ is zero if and only if A has rank $\leqslant n-2$.*

Problem 15 *Which of the following matrices are similar as matrices over $\mathbb{R}$?*

$$(a)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad (b)\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\quad (c)\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix},$$

$$(d)\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix},\quad (e)\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\quad (f)\begin{pmatrix}0&1&1\\0&1&0\\1&0&0\end{pmatrix}.$$

Problem 16 *For which $z \in \mathbb{C}$ does*

$$\sum_{n=0}^{\infty}\left(\frac{z^n}{n!} + \frac{n^2}{z^n}\right)$$

converge?

Problem 17 *Let P and Q be complex polynomials with the degree of Q at least two more than the degree of P. Prove there is an $r > 0$ such that if C is a closed curve outside $|z| = r$, then*

$$\int_C \frac{P(z)}{Q(z)}\, dz = 0.$$

Problem 18 *Show that for any continuous function $f : [0,1] \to \mathbb{R}$ and $\varepsilon > 0$, there is a function of the form*

$$g(x) = \sum_{k=0}^{n} C_k x^{4k}$$

for some $n \in \mathbb{Z}$, where $C_0, \ldots, C_n \in \mathbb{Q}$ and $|g(x) - f(x)| < \varepsilon$ for all x in $[0,1]$.

Problem 19 *Let P be a $n \times n$ real matrix such that $x^t P y = -y^t P x$ for all column vectors x, y in $\mathbb{R}^n$. Prove that P is skew-symmetric.*

Problem 20 *Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ having all three of the following properties:*

- *$f(x) = 0$ for $x < 0$ and $x > 2$,*
- *$f'(1) = 1$,*
- *f has derivatives of all orders.*