Preliminary Exam - Spring 1978

Problem 1 *Let $k \geqslant 0$ be an integer and define a sequence of maps*

$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = \frac{x^k}{x^2+n}, \qquad n = 1, 2, \ldots .$$

For which values of k does the sequence converge uniformly on $\mathbb{R}$? On every bounded subset of $\mathbb{R}$?

Problem 2 *Prove that a map $g : \mathbb{R}^n \to \mathbb{R}^n$ is continuous only if its graph is closed in $\mathbb{R}^n \times \mathbb{R}^n$. Is the converse true?*
Note: See also Problem **??**.

Problem 3 *Let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.*

Problem 4 *Evaluate*

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx\,.$$

Problem 5 *let $\mathbb{Z}_n$ denote the ring of integers modulo n. Let $\mathbb{Z}_n[x]$ be the ring of polynomials with coefficients in $\mathbb{Z}_n$. Let $\mathfrak{I}$ denote the ideal in $\mathbb{Z}_n[x]$ generated by $x^2 + x + 1$.*

1. *For which values of n, $1 \leqslant n \leqslant 10$, is the quotient ring $\mathbb{Z}_n[x]/\mathfrak{I}$ a field?*
2. *Give the multiplication table for $\mathbb{Z}_2/\mathfrak{I}$.*

Problem 6 *Prove that the sum of two algebraic numbers is algebraic. (An algebraic number is a complex number which is a root of a polynomial with rational coefficients.)*

Problem 7 *What is the volume enclosed by the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

Problem 8 *Consider the differential equation*

$$\frac{dx}{dt} = x^2 + t^2, \quad x(0) = 1.$$

1. *Prove that for some* $b > 0$*, there is a solution defined for* $t \in [0, b]$.

2. *Find an explicit value of* b *having the property in Part 1.*

3. *Find a* $c > 0$ *such that there is no solution on* $[0, c]$.

Problem 9 *Determine the Jordan Canonical Form of the matrix*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 4 \end{pmatrix}.$$

Problem 10 *Suppose* A *is a real* $n \times n$ *matrix.*

1. *Is it true that* A *must commute with its transpose?*

2. *Suppose the columns of* A *(considered as vectors) form an orthonormal set; is it true that the rows of* A *must also form an orthonormal set?*

Problem 11 *Show that there is a complex analytic function defined on the set* $U = \{z \in \mathbb{C} \mid |z| > 4\}$ *whose derivative is*

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Is there a complex analytic function on U *whose derivative is*

$$\frac{z^2}{(z-1)(z-2)(z-3)} \ ?$$

Problem 12 *Prove that the uniform limit of a sequence of complex analytic functions is complex analytic. Is the analogous theorem true for real analytic functions?*

Problem 13

1. *For which real numbers* $\alpha > 0$ *does the differential equation*

   $$\frac{dx}{dt} = x^\alpha, \quad x(0) = 0\,,$$

   have a solution on some interval $[0, b]$, $b > 0$*?*

2. *For which values of α are there intervals on which two solutions are defined?*

Problem 14 *Let G be a group of order 10 which has a normal subgroup of order 2. Prove that G is abelian.*

Problem 15 *Is $x^4 + 1$ irreducible over the field of real numbers? The field of rational numbers? A field with 16 elements?*

Problem 16 *Let A and B denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that A and B have a common eigenvector in $\mathbb{R}^n$.*

Problem 17 *Evaluate*

$$\int\int_{\mathcal{A}} e^{-x^2-y^2}\, dxdy\,,$$

where $\mathcal{A} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant 1\}$.

Problem 18 *Let M be a matrix with entries in a field $\mathbf{F}$. The* row rank *of M over $\mathbf{F}$ is the maximal number of rows which are linearly independent (as vectors) over $\mathbf{F}$. The* column rank *is similarly defined using columns instead of rows.*

1. *Prove row rank = column rank.*

2. *Find a maximal linearly independent set of columns of*

$$\begin{pmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & -4 & 4 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

   taking $\mathbf{F} = \mathbb{R}$.

3. *If $\mathbf{F}$ is a subfield of $\mathbf{K}$, and M has entries in $\mathbf{F}$, how is the row rank of M over $\mathbf{F}$ related to the row rank of M over $\mathbf{K}$?*

Problem 19 *Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable over $[b, 1]$ for all b such that $0 < b \leqslant 1$.*

1. *If f is bounded, prove that f is Riemann integrable over $[0, 1]$.*

2. *What if f is not bounded?*

Problem 20 *Consider the system of equations*

$$\begin{aligned} 3x + y - z + u^4 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

1. *Prove that for some* $\varepsilon > 0$*, the system can be solved for* (x, y, u) *as a function of* $z \in [-\varepsilon, \varepsilon]$*, with* $x(0) = y(0) = u(0) = 0$*. Are such functions* $x(z)$*,* $y(z)$ *and* $u(z)$ *continuous? Differentiable? Unique?*

2. *Show that the system cannot be solved for* (x, y, z) *as a function of*$u \in [-\delta, \delta]$*, for all* $\delta > 0$*.*