Problem 1 Let \( k \geq 0 \) be an integer and define a sequence of maps
\[
f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = \frac{x^k}{x^2 + n}, \quad n = 1, 2, \ldots.
\]
For which values of \( k \) does the sequence converge uniformly on \( \mathbb{R} \)? On every bounded subset of \( \mathbb{R} \)?

Problem 2 Prove that a map \( g : \mathbb{R}^n \to \mathbb{R}^n \) is continuous only if its graph is closed in \( \mathbb{R}^n \times \mathbb{R}^n \). Is the converse true?

Note: See also Problem ??.

Problem 3 Let \( f : \mathbb{C} \to \mathbb{C} \) be a nonconstant entire function. Prove that \( f(\mathbb{C}) \) is dense in \( \mathbb{C} \).

Problem 4 Evaluate
\[
\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} \, dx.
\]

Problem 5 let \( \mathbb{Z}_n \) denote the ring of integers modulo \( n \). Let \( \mathbb{Z}_n[x] \) be the ring of polynomials with coefficients in \( \mathbb{Z}_n \). Let \( \mathcal{I} \) denote the ideal in \( \mathbb{Z}_n[x] \) generated by \( x^2 + x + 1 \).

1. For which values of \( n \), \( 1 \leq n \leq 10 \), is the quotient ring \( \mathbb{Z}_n[x]/\mathcal{I} \) a field?

2. Give the multiplication table for \( \mathbb{Z}_2/\mathcal{I} \).

Problem 6 Prove that the sum of two algebraic numbers is algebraic. (An algebraic number is a complex number which is a root of a polynomial with rational coefficients.)

Problem 7 What is the volume enclosed by the ellipsoid
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?
\]
Problem 8  Consider the differential equation
\[
\frac{dx}{dt} = x^2 + t^2, \quad x(0) = 1.
\]

1. Prove that for some \( b > 0 \), there is a solution defined for \( t \in [0, b] \).

2. Find an explicit value of \( b \) having the property in Part 1.

3. Find a \( c > 0 \) such that there is no solution on \( [0, c] \).

Problem 9  Determine the Jordan Canonical Form of the matrix
\[
A = \begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 4
\end{pmatrix}.
\]

Problem 10  Suppose \( A \) is a real \( n \times n \) matrix.

1. Is it true that \( A \) must commute with its transpose?

2. Suppose the columns of \( A \) (considered as vectors) form an orthonormal set; is it true that the rows of \( A \) must also form an orthonormal set?

Problem 11  Show that there is a complex analytic function defined on the set \( U = \{ z \in \mathbb{C} \mid |z| > 4 \} \) whose derivative is
\[
\frac{z}{(z-1)(z-2)(z-3)}.
\]
Is there a complex analytic function on \( U \) whose derivative is
\[
\frac{z^2}{(z-1)(z-2)(z-3)}?
\]

Problem 12  Prove that the uniform limit of a sequence of complex analytic functions is complex analytic. Is the analogous theorem true for real analytic functions?

Problem 13  \[ 1. \text{ For which real numbers } \alpha > 0 \text{ does the differential equation}
\]
\[
\frac{dx}{dt} = x^\alpha, \quad x(0) = 0,
\]
\[ \text{have a solution on some interval } [0, b], \ b > 0? \]
2. For which values of $\alpha$ are there intervals on which two solutions are defined?

**Problem 14**  Let $G$ be a group of order 10 which has a normal subgroup of order 2. Prove that $G$ is abelian.

**Problem 15**  Is $x^4 + 1$ irreducible over the field of real numbers? The field of rational numbers? A field with 16 elements?

**Problem 16**  Let $A$ and $B$ denote real $n \times n$ symmetric matrices such that $AB = BA$. Prove that $A$ and $B$ have a common eigenvector in $\mathbb{R}^n$.

**Problem 17**  Evaluate

$$\int \int_A e^{-x^2 - y^2} \, dx \, dy,$$

where $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$.

**Problem 18**  Let $M$ be a matrix with entries in a field $F$. The row rank of $M$ over $F$ is the maximal number of rows which are linearly independent (as vectors) over $F$. The column rank is similarly defined using columns instead of rows.

1. Prove row rank = column rank.

2. Find a maximal linearly independent set of columns of

$$\begin{pmatrix}
1 & 0 & 3 & -2 \\
2 & 1 & 2 & 0 \\
0 & 1 & -4 & 4 \\
1 & 1 & 1 & 2 \\
1 & 0 & 1 & 2
\end{pmatrix}$$

taking $F = \mathbb{R}$.

3. If $F$ is a subfield of $K$, and $M$ has entries in $F$, how is the row rank of $M$ over $F$ related to the row rank of $M$ over $K$?

**Problem 19**  Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable over $[b, 1]$ for all $b$ such that $0 < b \leq 1$.

1. If $f$ is bounded, prove that $f$ is Riemann integrable over $[0, 1]$. 

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2. What if \( f \) is not bounded?

**Problem 20** Consider the system of equations

\[
\begin{align*}
3x + y - z + u^4 &= 0 \\
x - y + 2z + u &= 0 \\
2x + 2y - 3z + 2u &= 0
\end{align*}
\]

1. Prove that for some \( \varepsilon > 0 \), the system can be solved for \((x, y, u)\) as a function of \( z \in [-\varepsilon, \varepsilon] \), with \( x(0) = y(0) = u(0) = 0 \). Are such functions \( x(z) \), \( y(z) \) and \( u(z) \) continuous? Differentiable? Unique?

2. Show that the system cannot be solved for \((x, y, z)\) as a function of \( u \in [-\delta, \delta] \), for all \( \delta > 0 \).