Problem 1 Suppose $f$ is a differentiable function from the reals into the reals. Suppose $f'(x) > f(x)$ for all $x \in \mathbb{R}$, and $f(x_0) = 0$. Prove that $f(x) > 0$ for all $x > x_0$.

Problem 2 Suppose that $f$ is a real valued function of one real variable such that

$$\lim_{x \to c} f(x)$$

exists for all $c \in [a, b]$. Show that $f$ is Riemann integrable on $[a, b]$.

Problem 3 1. Evaluate $P_{n-1}(1)$, where $P_{n-1}(x)$ is the polynomial

$$P_{n-1}(x) = \frac{x^n - 1}{x - 1}.$$

2. Consider a circle of radius 1, and let $Q_1, Q_2, \ldots, Q_n$ be the vertices of a regular $n$-gon inscribed in the circle. Join $Q_1$ to $Q_2, Q_3, \ldots, Q_n$ by segments of a straight line. You obtain $(n - 1)$ segments of lengths $\lambda_2, \lambda_3, \ldots, \lambda_n$. Show that

$$\prod_{i=2}^{n} \lambda_i = n.$$
2. Find the center $Z$ of $G$; that is, the set of all elements $z$ of $G$ such that $az = za$ for all $a \in G$.

3. Show that the set $O$ of real orthogonal matrices is a subgroup of $G$ (a matrix is orthogonal if $AA^t = I$, where $A^t$ denotes the transpose of $A$). Show by example that $O$ is not a normal subgroup.

4. Find a nontrivial homomorphism from $G$ onto an abelian group.

Problem 7 A matrix of the form
\[
\begin{pmatrix}
1 & a_0 & a_0^2 & \ldots & a_0^n \\
1 & a_1 & a_1^2 & \ldots & a_1^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a_n & a_n^2 & \ldots & a_n^n
\end{pmatrix}
\]
where the $a_i$ are complex numbers, is called a Vandermonde matrix.

1. Prove that the Vandermonde matrix is invertible if $a_0, a_1, \ldots, a_n$ are all different.

2. If $a_0, a_1, \ldots, a_n$ are all different, and $b_0, b_1, \ldots, b_n$ are complex numbers, prove that there is a unique polynomial $f$ of degree $n$ with complex coefficients such that $f(a_0) = b_0$, $f(a_1) = b_1, \ldots, f(a_n) = b_n$.

Problem 8 Find a list of real matrices, as long as possible, such that

- the characteristic polynomial of each matrix is $(x - 1)^5(x + 1)$,
- the minimal polynomial of each matrix is $(x - 1)^2(x + 1)$,
- no two matrices in the list are similar to each other.

Problem 9 Find the solution of the differential equation
\[y'' - 2y' + y = 0,\]
subject to the conditions
\[y(0) = 1, \quad y'(0) = 1.\]
**Problem 10** In $\mathbb{R}^2$, consider the region $A$ defined by $x^2 + y^2 > 1$. Find differentiable real valued functions $f$ and $g$ on $A$ such that $\partial f/\partial x = \partial g/\partial y$ but there is no real valued function $h$ on $A$ such that $f = \partial h/\partial y$ and $g = \partial h/\partial x$.

**Problem 11** Let the sequence $a_0, a_1, \ldots$ be defined by the equation

$$1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} a_n(x - 3)^n \quad (0 < x < 1).$$

Find

$$\lim_{n \to \infty} \left( |a_n|^{\frac{1}{n}} \right).$$

**Problem 12** Let $p$ be an odd prime. Let $Q(p)$ be the set of integers $a$, $0 \leq a \leq p - 1$, for which the congruence

$$x^2 \equiv a \pmod{p}$$

has a solution. Show that $Q(p)$ has cardinality $(p + 1)/2$.

**Problem 13** Consider the family of square matrices $A(\theta)$ defined by the solution of the matrix differential equation

$$\frac{dA(\theta)}{d\theta} = BA(\theta)$$

with the initial condition $A(0) = I$, where $B$ is a constant square matrix.

1. Find a property of $B$ which is necessary and sufficient for $A(\theta)$ to be orthogonal for all $\theta$; that is, $A(\theta)^t = A(\theta)^{-1}$, where $A(\theta)^t$ denotes the transpose of $A(\theta)$.

2. Find the matrices $A(\theta)$ corresponding to

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and give a geometric interpretation.

**Problem 14** A square matrix $A$ is nilpotent if $A^k = 0$ for some positive integer $k$. 

3
1. If $A$ and $B$ are nilpotent, is $A + B$ nilpotent?

2. Prove: If $A$ and $B$ are nilpotent matrices and $AB = BA$, then $A + B$ is nilpotent.

3. Prove: If $A$ is nilpotent then $I + A$ and $I - A$ are invertible.

**Problem 15** Let $f(z)$ be a nonconstant meromorphic function. A complex number $w$ is called a period of $f$ if $f(z + w) = f(z)$ for all $z$.

1. Show that if $w_1$ and $w_2$ are periods, so are $n_1 w_1 + n_2 w_2$ for all integers $n_1$ and $n_2$.

2. Show that there are, at most, a finite number of periods of $f$ in any bounded region of the complex plane.

**Problem 16** Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos nx}{x^4 + 1} \, dx.$$ 

**Problem 17** Let $\mathbb{Q}_+$ be the multiplicative group of positive rational numbers.

1. Is $\mathbb{Q}_+$ torsion free?

2. Is $\mathbb{Q}_+$ free?

**Problem 18** 1. In $\mathbb{R}[x]$, consider the set of polynomials $f(x)$ for which $f(2) = f'(2) = f''(2) = 0$. Prove that this set forms an ideal and find its monic generator.

2. Do the polynomials such that $f(2) = 0$ and $f'(3) = 0$ form an ideal?

**Problem 19** Suppose that $u(x, t)$ is a continuous function of the real variables $x$ and $t$ with continuous second partial derivatives. Suppose that $u$ and its first partial derivatives are periodic in $x$ with period 1, and that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

Prove that

$$E(t) = \frac{1}{2} \int_0^1 \left( \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \right) \, dx$$

is a constant independent of $t$. 

Problem 20  Let \( h : [0, 1) \to \mathbb{R} \) be a function defined on the half-open interval \([0, 1)\). Prove that if \( h \) is uniformly continuous, there exists a unique continuous function \( g : [0, 1] \to \mathbb{R} \) such that \( g(x) = h(x) \) for all \( x \in [0, 1) \).