Problem 1 Let $F$ be a finite field of order $q$, and let $V$ be a two dimensional vector space over $F$. Find the number of endomorphisms of $V$ that fix at least one nonzero vector.

Problem 2 Let the continuous function $f : \mathbb{R} \to \mathbb{R}$ be periodic with period 1. Prove that, for any real number $c$, there is a real number $x_0$ such that $f(x_0 + c) = f(x_0)$.

Problem 3 Find all commutative rings $R$ with identity such that $R$ has a unique maximal ideal and such that the group of units of $R$ is trivial.

Problem 4 Evaluate $$\int_{0}^{\infty} \frac{1}{1+x^5} dx.$$ 

Problem 5 Let $\alpha$ and $\beta$ be real numbers such that the subgroup $\Gamma$ of $\mathbb{R}$ generated by $\alpha$ and $\beta$ is closed. Prove that $\alpha$ and $\beta$ are linearly dependent over $\mathbb{Q}$.

Problem 6 Let $S$ be a special orthogonal $n \times n$ matrix, a real $n \times n$ matrix satisfying $S^t S = I$ and $\det(S) = 1$.

1. Prove that if $n$ is odd then 1 is an eigenvalue of $S$.

2. Prove that if $n$ is even then 1 need not be an eigenvalue of $S$.

Problem 7 Let the functions $f_n : [0,1] \to [0,1]$ $(n = 1,2,\ldots)$ satisfy $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq 1/n$. Prove that the sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.

Problem 8 If $G$ is a finite group, must $S = \{g^2 \mid g \in G\}$ be a subgroup? Provide a proof or a counterexample.

Problem 9 1. Prove that an entire function with a positive real part is constant.
2. Prove the analogous result for $2 \times 2$ matrix functions: If $F(z) = (f_{jk}(z))$ is a matrix function in the complex plane, each entry $f_{jk}$ being entire, and if $F(z) + F(z)^*$ is positive definite for each $z$, then $F$ is constant. (Here, $F(z)^*$ is the conjugate transpose of $F(z)$.)

**Problem 10** Let $M_n$ be the vector space of $n \times n$ complex matrices. For $A$ in $M_n$ define the linear transformation of $T_A$ on $M_n$ by $T_A(X) = AX - XA$. Prove that the rank of $T_A$ is at most $n^2 - n$.

**Problem 11** Let $U$ be a nonempty, proper, open subset of $\mathbb{R}^n$. Construct a function $f : \mathbb{R}^n \to \mathbb{R}$ that is discontinuous at each point of $U$ and continuous at each point of $\mathbb{R}^n \setminus U$.

**Problem 12** Let $S_9$ denote the group of permutations of $\{1, 2, \ldots, 9\}$, and let $A_9$ denote the group of even permutations. Let $1$ denote the identity permutation in $S_9$.

1. Determine the smallest positive integer $m$ such that $\sigma^m = 1$ for all $\sigma$ in $S_9$.

2. Determine the smallest positive integer $n$ such that $\sigma^n = 1$ for all $\sigma$ in $A_9$.

**Problem 13** Let the power series $\sum_{n=0}^{\infty} c_n z^n$, with positive radius of convergence $R$, represent the function $f$ in the disk $|z| < R$. For $k = 0, 1, \ldots$ let $s_k$ be the $k$-th partial sum of the series $s_k(z) = \sum_{n=0}^{k} c_n z^n$. Prove that $\sum_{k=0}^{\infty} |f(z) - s_k(z)| < \infty$ for each $z$ in the disk $|z| < R$.

**Problem 14** Consider the differential-delay equation given by $y'(t) = -y(t-t_0)$. Here, the independent variable $t$ is a real variable, the function $y$ is allowed to be complex valued, and $t_0$ is a positive constant. Prove that if $0 < t_0 < \pi/2$ then every solution of the form $y(t) = e^{\lambda t}$, with $\lambda$ complex, tends to 0 as $t \to +\infty$.

**Problem 15** Let $A$ be an $n \times n$ matrix over a field $K$. Prove that

$$\text{rank } A^2 - \text{rank } A^3 \leq \text{rank } A - \text{rank } A^2$$
Problem 16 Let $T_0$ be the interior of a triangle in $\mathbb{R}^2$ with vertices $A, B, C$. Let $T_1$ be the interior of the triangle whose vertices are the midpoints of the sides of $T_0$, $T_2$ the interior of the triangle whose vertices are the midpoints of the sides of $T_1$, and so on. Describe the set $\cap_{n=0}^{\infty} T_n$.

Problem 17 Prove that the polynomial $f(x) = 16x^5 - 125x^4 + 50x^3 - 100x^2 + 75x + 25$ is irreducible over the rationals.

Problem 18 Let $f$ be an entire function such that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq Ar^{2k} \quad (0 < r < \infty),$$

where $k$ is a positive integer and $A$ is a positive constant. Prove that $f$ is a constant multiple of the function $z^k$. 