Preliminary Exam - Spring 2001

Problem 1 *Let* $\mathbf{F}$ *be a finite field of order* q*, and let* V *be a two dimensional vector space over* $\mathbf{F}$*. Find the number of endomorphisms of* V *that fix at least one nonzero vector.*

Problem 2 *Let the continuous function* $f : \mathbb{R} \to \mathbb{R}$ *be periodic with period 1. Prove that, for any real number* c*, there is a real number* x_0 *such that* $f(x_0 + c) = f(x_0)$*.*

Problem 3 *Find all commutative rings* R *with identity such that* R *has a unique maximal ideal and such that the group of units of* R *is trivial.*

Problem 4 *Evaluate*

$$\int_0^\infty \frac{1}{1+x^5} dx\,.$$

Problem 5 *Let* α *and* β *be real numbers such that the subgroup* Γ *of* $\mathbb{R}$ *generated by* α *and* β *is closed. Prove that* α *and* β *are linearly dependent over* $\mathbb{Q}$ *.*

Problem 6 *Let* S *be a* special orthogonal $n \times n$ *matrix, a real* $n \times n$ *matrix satisfying* $S^t S = I$ *and* $det(S) = 1$.

1. *Prove that if* n *is odd then 1 is an eigenvalue of* S*.*
2. *Prove that if* n *is even then 1 need not be an eigenvalue of* S*.*

Problem 7 *Let the functions* $f_n : [0,1] \to [0,1]$ $(n = 1, 2, \ldots)$ *satisfy* $|f_n(x) - f_n(y)| \leqslant |x-y|$ *whenever* $|x-y| \geqslant 1/n$*. Prove that the sequence* $\{f_n\}_{n=1}^\infty$ *has a uniformly convergent subsequence.*

Problem 8 *If* G *is a finite group, must* $S = \{g^2 \,|\, g \in G\}$ *be a subgroup? Provide a proof or a counterexample.*

Problem 9

1. *Prove that an entire function with a positive real part is constant.*

2. *Prove the analogous result for* 2×2 *matrix functions: If* $F(z) = (f_{jk}(z))$ *is a matrix function in the complex plane, each entry* f_{jk} *being entire, and if* $F(z) + F(z)^*$ *is positive definite for each* z*, then* F *is constant. (Here,* $F(z)^*$ *is the conjugate transpose of* $F(z)$*.)*

Problem 10 *Let* M_n *be the vector space of* $n \times n$ *complex matrices. For* A *in* M_n *define the linear transformation of* T_A *on* M_n *by* $T_A(X) = AX - XA$*. Prove that the rank of* T_A *is at most* $n^2 - n$*.*

Problem 11 *Let* U *be a nonempty, proper, open subset of* $\mathbb{R}^n$*. Construct a function* $f : \mathbb{R}^n \to \mathbb{R}$ *that is discontinuous at each point of* U *and continuous at each point of* $\mathbb{R}^n \backslash U$*.*

Problem 12 *Let* S_9 *denote the group of permutations of* $\{1, 2, \ldots, 9\}$*, and let* A_9 *denote the group of even permutations. Let 1 denote the identity permutation in* S_9*.*

1. *Determine the smallest positive integer* m *such that* $\sigma^m = 1$ *for all* σ *in* S_9*.*

2. *Determine the smallest positive integer* n *such that* $\sigma^n = 1$ *for all* σ *in* A_9*.*

Problem 13 *Let the power series* $\sum_{n=0}^{\infty} c_n z^n$*, with positive radius of convergence* R*, represent the function* f *in the disk* $|z| < R$*. For* $k = 0, 1, \ldots$ *let* s_k *be the* k*-th partial sum of the series* $s_k(z) = \sum_{n=0}^{k} c_n z^n$*. Prove that*

$$\sum_{k=0}^{\infty} |f(z) - s_k(z)| < \infty$$

for each z *in the disk* $|z| < R$*.*

Problem 14 *Consider the* differential-delay equation *given by* $y'(t) = -y(t - t_0)$*. Here, the independent variable* t *is a real variable, the function* y *is allowed to be complex valued, and* t_0 *is a positive constant. Prove that if* $0 < t_0 < \pi/2$ *then every solution of the form* $y(t) = e^{\lambda t}$*, with* λ *complex, tends to 0 as* $t \to +\infty$*.*

Problem 15 *Let* A *be an* $n \times n$ *matrix over a field* K*. Prove that*

$$\text{rank } A^2 - \text{rank } A^3 \leqslant \text{rank } A - \text{rank } A^2$$

Problem 16 *Let* T_0 *be the interior of a triangle in* $\mathbb{R}^2$ *with vertices* A, B, C*. Let* T_1 *be the interior of the triangle whose vertices are the midpoints of the sides of* T_0*,* T_2 *the interior of the triangle whose vertices are the midpoints of the sides of* T_1*, and so on. Describe the set* $\cap_{n=0}^{\infty} T_n$ *.*

Problem 17 *Prove that the polynomial* $f(x) = 16x^5 - 125x^4 + 50x^3 - 100x^2 + 75x + 25$ *is irreducible over the rationals.*

Problem 18 *Let* f *be an entire function such that*

$$\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \ \leqslant \ Ar^{2k} \qquad (0 < r < \infty) \ ,$$

where k *is a positive integer and* A *is a positive constant. Prove that* f *is a constant multiple of the function* z^k*.*