Problem 1  Are the $4 \times 4$ matrices
\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
-1 & 1 & 1 & -1 \\
-1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0
\end{pmatrix}
\]
similar?

Problem 2  Let \( \{f_n\}_{n=1}^{\infty} \) be a uniformly bounded equicontinuous sequence of real-valued functions on the compact metric space \((X, d)\). Define the functions \( g_n : X \to \mathbb{R} \), for \( n \in \mathbb{N} \) by
\[
g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}.
\]
Prove that the sequence \( \{g_n\}_{n=1}^{\infty} \) converges uniformly.

Problem 3  Prove that the group \( G = \mathbb{Q}/\mathbb{Z} \) has no proper subgroup of finite index.

Problem 4  Let \( f \) and \( g \) be entire functions such that \( \lim_{z \to \infty} f(g(z)) = \infty \).
Prove that \( f \) and \( g \) are polynomials.

Problem 5  Let \( a \) and \( x_0 \) be positive numbers, and define the sequence \( (x_n)_{n=1}^{\infty} \) recursively by
\[
x_n = \frac{1}{2} \left( x_{n-1} + \frac{a}{x_{n-1}} \right).
\]
Prove that this sequence converges, and find its limit.

Problem 6  Let \( A \) be an \( n \times n \) matrix over \( \mathbb{C} \) whose minimal polynomial \( \mu \) has degree \( k \).

1. Prove that, if the point \( \lambda \) of \( \mathbb{C} \) is not an eigenvalue of \( A \), then there is a polynomial \( p_\lambda \) of degree \( k - 1 \) such that \( p_\lambda(A) = (A - \lambda I)^{-1} \).
2. Let $\lambda_1, \ldots, \lambda_k$ be distinct points of $\mathbb{C}$ that are not eigenvalues of $A$. Prove that there are complex numbers $c_1, \ldots, c_k$ such that
$$\sum_{j=1}^{k} c_j (A - \lambda_j I)^{-1} = I.$$ 

**Problem 7** Let $f$ be a positive function of class $C^2$ on $(0, \infty)$ such that $f' \leq 0$ and $f''$ is bounded. Prove that $\lim_{t \to \infty} f'(t) = 0$.

**Problem 8** Find the cardinality of the set of all subrings of $\mathbb{Q}$, the field of rational numbers.

**Problem 9** Evaluate
$$I = \int_{|z|=1} \frac{\cos^3 z}{z^3} \, dz,$$
where the direction of integration is counterclockwise.

**Problem 10** Let $S$ be an uncountable subset of $\mathbb{R}$. Prove that there exists a real number $t$ such that both sets $S \cap (-\infty, t)$ and $S \cap (t, \infty)$ are uncountable.

**Problem 11** Let $A_n$ be the $n \times n$ matrix whose entries $a_{jk}$ are given by
$$a_{jk} = \begin{cases} 1 & \text{if } |j - k| = 1 \\ 0 & \text{otherwise} \end{cases}.$$ 
Prove that the eigenvalues of $A$ are symmetric with respect to the origin.

**Problem 12** Suppose that $H_1$ and $H_2$ are distinct subgroups of a group $G$ such that $[G : H_1] = [G : H_2] = 3$. What are the possible values of $[G : H_1 \cap H_2]$?

**Problem 13** Let $f$ be a nonconstant entire function whose values on the real axis are real and nonnegative. Prove that all real zeros of $f$ have even order.

**Problem 14** Let $I_1, \ldots, I_n$ be disjoint closed nonempty subintervals of $\mathbb{R}$. 

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1. Prove that if \( p \) is a real polynomial of degree less than \( n \) such that
\[
\int_{I_j} p(x)dx = 0, \quad \text{for } j = 1, \ldots, n
\]
then \( p = 0 \).

2. Prove that there is a nonzero real polynomial \( p \) of degree \( n \) that satisfies all the above equations.

**Problem 15** Let \( F \), with components \( F_1, \ldots, F_n \), be a differentiable map of \( \mathbb{R}^n \) into \( \mathbb{R}^n \) such that \( F(0) = 0 \). Assume that
\[
\sum_{j,k=1}^n \left| \frac{\partial F_j(0)}{\partial x_k} \right|^2 = c < 1 .
\]
Prove that there is a ball \( B \) in \( \mathbb{R}^n \) with center 0 such that \( F(B) \subset B \).

**Problem 16** Let \( A \) be a complex \( n \times n \) matrix such that the sequence \( (A^n)_{n=1}^\infty \) converges to a matrix \( B \). Prove that \( B \) is similar to a diagonal matrix with zeros and ones along the main diagonal.

**Problem 17** Evaluate the integrals
\[
I(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{(x+i)^2} \, dx , \quad -\infty < t < \infty .
\]

**Problem 18** Let \( G \) be a finite group and \( p \) a prime number. Suppose \( a \) and \( b \) are elements of \( G \) of order \( p \) such that \( b \) is not in the subgroup generated by \( a \). Prove that \( G \) contains at least \( p^2 - 1 \) elements of order \( p \).