Preliminary Exam - Spring 2000

Problem 1 *Are the 4×4 matrices*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

similar?

Problem 2 *Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded equicontinuous sequence of real-valued functions on the compact metric space (X, d). Define the functions $g_n : X \to \mathbb{R}$, for $n \in \mathbb{N}$ by*

$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}\,.$$

Prove that the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly.

Problem 3 *Prove that the group $G = \mathbb{Q}/\mathbb{Z}$ has no proper subgroup of finite index.*

Problem 4 *Let f and g be entire functions such that $\lim_{z\to\infty} f(g(z)) = \infty$. Prove that f and g are polynomials.*

Problem 5 *Let a and x_0 be positive numbers, and define the sequence $(x_n)_{n=1}^{\infty}$ recursively by*

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{a}{x_{n-1}}\right)\,.$$

Prove that this sequence converges, and find its limit.

Problem 6 *Let A be an $n \times n$ matrix over $\mathbb{C}$ whose minimal polynomial μ has degree k.*

1. *Prove that, if the point λ of $\mathbb{C}$ is not an eigenvalue of A, then there is a polynomial p_λ of degree $k-1$ such that $p_\lambda(A) = (A - \lambda I)^{-1}$.*

2. *Let $\lambda_1, \dots, \lambda_k$ be distinct points of $\mathbb{C}$ that are not eigenvalues of A. Prove that there are complex numbers $c_1, \dots, c_k$ such that*

$$\sum_{j=1}^{k} c_j (A - \lambda_j I)^{-1} = I\,.$$

Problem 7 *Let f be a positive function of class C^2 on $(0, \infty)$ such that $f' \leqslant 0$ and f'' is bounded. Prove that $\lim_{t \to \infty} f'(t) = 0$.*

Problem 8 *Find the cardinality of the set of all subrings of $\mathbb{Q}$, the field of rational numbers.*

Problem 9 *Evaluate*

$$I = \int_{|z|=1} \frac{\cos^3 z}{z^3}\, dz\,,$$

where the direction of integration is counterclockwise.

Problem 10 *Let S be an uncountable subset of $\mathbb{R}$. Prove that there exists a real number t such that both sets $S \cap (-\infty, t)$ and $S \cap (t, \infty)$ are uncountable.*

Problem 11 *Let A_n be the $n \times n$ matrix whose entries a_{jk} are given by*

$$a_{jk} = \begin{cases} 1 & \text{if} \quad |j-k| = 1 \\ 0 & \quad\text{otherwise}\,. \end{cases}$$

Prove that the eigenvalues of A are symmetric with respect to the origin.

Problem 12 *Suppose that H_1 and H_2 are distinct subgroups of a group G such that $[G : H_1] = [G : H_2] = 3$. What are the possible values of $[G : H_1 \cap H_2]$?*

Problem 13 *Let f be a nonconstant entire function whose values on the real axis are real and nonnegative. Prove that all real zeros of f have even order.*

Problem 14 *Let $I_1, \dots, I_n$ be disjoint closed nonempty subintervals of $\mathbb{R}$.*

1. *Prove that if p is a real polynomial of degree less than n such that*

$$\int_{I_j} p(x)dx = 0 \ , \qquad \text{for } j = 1, \ldots, n$$

then $p = 0$.

2. *Prove that there is a nonzero real polynomial p of degree n that satisfies all the above equations.*

Problem 15 *Let F, with components $F_1, \ldots, F_n$, be a differentiable map of $\mathbb{R}^n$ into $\mathbb{R}^n$ such that $F(0) = 0$. Assume that*

$$\sum_{j,k=1}^{n} \left| \frac{\partial F_j(0)}{\partial x_k} \right|^2 = c < 1 \ .$$

Prove that there is a ball B in $\mathbb{R}^n$ with center 0 such that $F(B) \subset B$.

Problem 16 *Let A be a complex $n \times n$ matrix such that the sequence $(A^n)_{n=1}^{\infty}$ converges to a matrix B. Prove that B is similar to a diagonal matrix with zeros and ones along the main diagonal.*

Problem 17 *Evaluate the integrals*

$$I(t) = \int_{-\infty}^{\infty} \frac{e^{itx}}{(x+i)^2} \ dx \ , \qquad -\infty < t < \infty \ .$$

Problem 18 *Let G be a finite group and p a prime number. Suppose a and b are elements of G of order p such that b is not in the subgroup generated by a. Prove that G contains at least $p^2 - 1$ elements of order p.*