1. Please write your 1- or 2-digit exam number on this cover sheet and on all problem sheets (even problems that you do not wish to be graded).

2. Indicate below which six problems you wish to have graded. Cross out solutions you may have begun for the problems that you have not selected.

3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem $p$ on either side of the page for problem $q$ if $p \neq q$.

4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

- 1A. ______ 1B. ______ Calculus
- 2A. ______ 2B. ______ Real analysis
- 3A. ______ 3B. ______ Real analysis
- 4A. ______ 4B. ______ Complex analysis
- 5A. ______ 5B. ______ Complex analysis
- 6A. ______ 6B. ______ Linear algebra
- 7A. ______ 7B. ______ Linear algebra
- 8A. ______ 8B. ______ Abstract algebra
- 9A. ______ 9B. ______ Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______
Problem 1A. 

Let \( y \) be a solution of \( y''' - y = 0 \) such that \( y(t) \to 0 \) as \( t \to \infty \). Show that \( y(0) + y'(0) + y''(0) = 0 \).

Solution:
Problem 2A. 

Let \( f : \mathbb{R} \to \mathbb{R} \) be bounded and continuously differentiable. Show that every solution \( y \) of \( y' = f(y) \) is monotone.

Solution:
Problem 3A.

Let $f$ be a twice continuously differentiable function on $[0, 1]$ such that $f(0) = f(1) = 0$. Prove that

$$\max_{x \in [0,1]} |f(x)| \leq \frac{1}{8} \max_{x \in [0,1]} |f''(x)|,$$

and find an example where equality holds.

Solution:
Problem 4A.  

Evaluate

\[ I = \int_{-\infty}^{\infty} \frac{x \sin x}{(1 + x^2)^2} \, dx. \]

Solution:
Problem 5A.

Find the number of complex roots of $e^z = 3z^6$ with $|z| < 1$ that have positive imaginary part.

Solution:
Let $n$ be a positive integer and let $a$ be a complex number. Prove that $a^n = 1$ if and only if there are invertible $n$ by $n$ complex matrices $X$, $Y$ such that $YX = aXY$.

Solution:
Problem 7A.

Let $A$ be a complex $n \times n$ matrix satisfying $A^{37} = I$. Show that $A$ is diagonalizable.

Solution:
Problem 8A.

Show that $F : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ defined by

$$F(u, v, w) = (-u - v - w, uv + uw + vw, -uvw)$$

is surjective but not injective.

Solution:
Let \( S \) be a countable set of real numbers. Show that there are function \( g_n : \mathbb{N} \to \mathbb{N} \) such that if \( f : \mathbb{N} \to \mathbb{N} \) is a function from \( \mathbb{N} \) to \( \mathbb{N} \) with \( f(n + 1) > g_n(f(n)) \) for all \( n \), then
\[
A = \sum_{n=1}^{\infty} \frac{1}{f(n)}
\]
converges to a real that is not in the set \( S \).

Solution:
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PROBLEM SELECTION

Part B: List the six problems you have chosen:
Problem 1B.

Evaluate

\[ I = \int_{0}^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \, dx \]

Solution:
Problem 2B.

For $t \geq 0$ let

$$F(t) = \int_0^t \exp(-x^2) \, dx$$

and

$$G(t) = \int_0^1 \frac{\exp(-t^2(1 + x^2))}{1 + x^2} \, dx.$$ 

Show that $F(t)^2 + G(t)$ is constant and deduce the value of $F(\infty)$.

Solution:
Show that \( x_{n+1} = (1 + x_n)^{-1} \) converges and find its limit for any \( x_0 > 0 \).

Solution:
Let $c_0, c_1, \ldots, c_{n-1}$ be complex numbers. Prove that all the zeroes of the polynomial

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$$

lie in the open disc with center 0 and radius

$$1 + |c_{n-1}| + \cdots + |c_1| + |c_0|.$$ 

**Solution:**
If $f(z)$ is analytic in the open disc $\mathbb{D} = \{ z : |z| < 1 \}$, and if $|f(z)| < 1/(1 - |z|)$ for all $z \in \mathbb{D}$, show that

$$\left| \frac{f^{(n)}(0)}{n!} \right| \leq (n + 1) \left( 1 + \frac{1}{n} \right)^n < e(n + 1).$$

Solution:
Let $\mathbb{Z}_2$ be the ring of integers mod 2. Prove the following identity in $\mathbb{Z}_2[x_1, \ldots, x_n]$:

$$\begin{vmatrix}
x_1 & \ldots & x_n \\
x_1^2 & \ldots & x_n^2 \\
x_1^{2^{n-1}} & \ldots & x_n^{2^{n-1}}
\end{vmatrix} = \prod_{(a_1, \ldots, a_n) \neq (0, \ldots, 0)} (a_1 x_1 + \cdots + a_n x_n),$$

where $(a_1 \ldots a_n)$ run all non-zero values in $\mathbb{Z}_2^n$.

Solution:
Is it true that elements of the group $GL^+_2(\mathbb{R})$ of real $2 \times 2$-matrices with positive determinant are conjugate in $GL^+_2(\mathbb{R})$ if and only if the matrices are similar (conjugate in $GL_2(\mathbb{R})$)? Either prove this or give a counterexample.

Solution:
Let $R$ be a ring (possibly non-commutative, possibly without an identity 1) in which every element is idempotent (this means that for all $a \in R$, $a^2 = a$). Show that $R$ has characteristic $2$ ($2a = 0$ for all $a$) and is commutative.

Solution:
Recall that $S_6$ and $A_6$ are the symmetric group and alternating group on 6 letters, respectively.

Prove or give a counterexample (with explanation): For every $\sigma \in A_6$ there is a $\tau \in S_6$ such that $\tau^2 = \sigma$.

Solution: