Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Spring Semester 2017

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION (for use by grader—do not write below)

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Show that the following improper Riemann integrals exist and are equal:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} dx$$

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Problem 2A. *Score:*

Suppose f is a function from the reals to the reals satisfying $2f(x) = f(2x)$ for all x.

(a) Prove that if f is differentiable at 0 then f is linear.

(b) Give an example of such a function f that is continuous but not linear.

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Problem 3A. *Score:*

Suppose we have a continuous positive function $f : (0, \pi) \to (0, \infty)$ such that for all $x, y \in (0, \pi)$ we have

$$\int_x^y \frac{f(x)f(y)}{f^2(t)} dt = \sin(y - x).$$

(a) Show that $\sin(z - x)f(y) = \sin(y - x)f(z) + \sin(z - y)f(x)$.

(b) Find all possibilities for f.

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Problem 4A. *Score:*

The Weierstrass zeta function ζ is a meromorphic function satisfying

- $\zeta(z+\omega_1) = \zeta(z) + \eta_1$
- $\zeta(z+\omega_2) = \zeta(z) + \eta_2$
- The singularities of ζ are poles of residue 1 at the points $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$

Here ω_1, ω_2, η_1, η_2 are complex constants with ω_2/ω_1 not real. Use Cauchy's residue theorem to prove Legendre's relation $\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi i$ and express the sign in terms of ω_1 and ω_2.

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Problem 5A. *Score:*

Suppose the coefficients of the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

are given by the recurrence relation

$$a_0 = 1,\ a_1 = -1,\ 3a_n + 4a_{n-1} - a_{n-2} = 0,\ n = 2, 3, \dots\ .$$

Find the radius of convergence of the series and the function to which it converges in its disc of convergence.

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Problem 6A. *Score:*

Let A be an $n \times n$ matrix over the complex numbers. Let $e^A = 1 + A + A^2/2 + \cdots + A^m/m! + \cdots$. Show this series converges and $\det(e^A) = e^{Tr(A)}$.

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Problem 7A. *Score:*

Given two vectors x and y in $\mathbb{R}^n$ with $\|x\|_2 = \|y\|_2$, construct an orthogonal matrix Q such that $Qx = y$. Can there be such a matrix if $\|x\|_2 \neq \|y\|_2$?

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Problem 8A. *Score:*

Show that for each integer $p \geq 0$ the sum

$$S_p(n) = \sum_{k=0}^{n} k^p$$

is a polynomial of degree $p + 1$ in the variable n.

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Problem 9A. *Score:*

The Bell number P_n is the number of partitions of a set of n elements into disjoint nonempty subsets, so for example $\{1,2,3\} = \{1\} \cup \{2\} \cup \{3\} = \{1,2\} \cup \{3\} = \{2,3\} \cup \{1\} = \{1,3\} \cup \{2\}$ and $P_3 = 5$. Show that

$$\frac{P_n}{n!} \to 0$$

as $n \to \infty$.

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GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2017

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

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Problem 1B. *Score:*

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

for all $x \in \mathbb{R}$ and all $h \neq 0$. (Hint: multiply both sides by $2h$.)

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Problem 2B. *Score:*

Suppose $f : [-1,1] \to \mathbb{C}$ is a continuous complex-valued function, and for all non-negative integers n

$$\int_{-1}^{1} x^n f(x)dx = 0.$$

Prove that $f = 0$.

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Problem 3B. *Score:*

The error of a quadrature rule with $p+1$ distinct points x_j, weights w_j is

$$E_p(f) = \int_a^b f(x)dx - \sum_{j=0}^{p} w_j f(x_j).$$

Suppose that $E_p(f) = 0$ whenever f is a polynomial of degree $\leq q$. Show that $q \leq 2p+1$ and if $q \geq 2p$ then $w_j > 0$ for all j.

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Problem 4B. *Score:*

Given n distinct points $z_j \in \mathbb{C}$ and n values $f_j \in \mathbb{C}$, show that there is a unique polynomial P of degree at most $n-1$ such that

$$P(z_j) = f_j$$

for $1 \leq j \leq n$.

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Problem 5B. *Score:*

Write all values of i^i in the form $a + bi$.

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Problem 6B. *Score:*

Let D be the unit disk in the complex plane $\mathbb{C}$, $f : D \to \mathbb{C}$ an analytic function with

$$|f^{(k)}(0)| \leq M$$

for all $k \geq 0$, and let $t_p \in D$, $s_p \in D$ for $1 \leq p \leq n$. For each $n \geq 1$ define $A_{ij} = f(t_i s_j)$ for $1 \leq i, j \leq n$. For each $r \geq 1$ find an $n \times n$ matrix B with rank $\leq r$ and

$$|A_{ij} - B_{ij}| \leq \frac{2M}{r!}$$

for $1 \leq i, j \leq n$.

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Problem 7B. *Score:*

Suppose R is an invertible upper triangular complex matrix and A is symmetric. Find an explicit formula for the entries of the upper triangular matrix E satisfying

$$E^T R + R^T E = A$$

and show that your solution is unique. Hint: Multiply by R^{-1T} and R^{-1}.

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Problem 8B. *Score:*

Find a product of cyclic groups of prime power order isomorphic to $(\mathbb{Z}/1000000\mathbb{Z})^*$ (the group of units of the ring of integers mod 1000000).

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Problem 9B. *Score:*

Let S_9 denote the group of permutations of 9 objects.

(a) Exhibit an element of S_9 of order 20.

(b) Prove that no element of S_9 has order 18.