1. Please write your 1- or 2-digit exam number on this cover sheet and on all problem sheets (even problems that you do not wish to be graded).

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3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem \( p \) on either side of the page for problem \( q \) if \( p \neq q \).

4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______, ______, ______, ______, ______, ______

GRADE COMPUTATION (for use by grader—do not write below)

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Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______
Show that the following improper Riemann integrals exist and are equal:

\[ \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \, dx = \int_{-\infty}^{\infty} \frac{\sin^2(x)}{x^2} \, dx \]
Suppose $f$ is a function from the reals to the reals satisfying $2f(x) = f(2x)$ for all $x$.

(a) Prove that if $f$ is differentiable at 0 then $f$ is linear.

(b) Give an example of such a function $f$ that is continuous but not linear.
Problem 3A.

Suppose we have a continuous positive function $f : (0, \pi) \rightarrow (0, \infty)$ such that for all $x, y \in (0, \pi)$ we have

$$\int_{x}^{y} \frac{f(x)f(y)}{f^2(t)} dt = \sin(y - x).$$

(a) Show that $\sin(z - x)f(y) = \sin(y - x)f(z) + \sin(z - y)f(x)$.

(b) Find all possibilities for $f$. 

The Weierstrass zeta function $\zeta$ is a meromorphic function satisfying

- $\zeta(z + \omega_1) = \zeta(z) + \eta_1$
- $\zeta(z + \omega_2) = \zeta(z) + \eta_2$
- The singularities of $\zeta$ are poles of residue 1 at the points $m\omega_1 + n\omega_2$ for $m, n \in \mathbb{Z}$

Here $\omega_1, \omega_2, \eta_1, \eta_2$ are complex constants with $\frac{\omega_2}{\omega_1}$ not real. Use Cauchy’s residue theorem to prove Legendre’s relation $\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi i$ and express the sign in terms of $\omega_1$ and $\omega_2$. 
Suppose the coefficients of the power series
\[ \sum_{n=0}^{\infty} a_n z^n \]
are given by the recurrence relation
\[ a_0 = 1, \ a_1 = -1, \ 3a_n + 4a_{n-1} - a_{n-2} = 0, \ n = 2, 3, \ldots \]
Find the radius of convergence of the series and the function to which it converges in its disc of convergence.
Problem 6A.

Let $A$ be an $n \times n$ matrix over the complex numbers. Let $e^A = 1 + A + A^2/2 + \cdots + A^m/m! + \cdots$. Show this series converges and $\det(e^A) = e^{\text{Tr}(A)}$. 
Given two vectors $x$ and $y$ in $\mathbb{R}^n$ with $\|x\|_2 = \|y\|_2$, construct an orthogonal matrix $Q$ such that $Qx = y$. Can there be such a matrix if $\|x\|_2 \neq \|y\|_2$?
Problem 8A.  

Show that for each integer $p \geq 0$ the sum

$$S_p(n) = \sum_{k=0}^{n} k^p$$

is a polynomial of degree $p + 1$ in the variable $n$. 
The Bell number $P_n$ is the number of partitions of a set of $n$ elements into disjoint nonempty subsets, so for example $\{1, 2, 3\} = \{1\} \cup \{2\} \cup \{3\} = \{1, 2\} \cup \{3\} = \{2, 3\} \cup \{1\} = \{1, 3\} \cup \{2\}$ and $P_3 = 5$. Show that

$$\frac{P_n}{n!} \to 0$$

as $n \to \infty$. 
YOUR 1 OR 2 DIGIT EXAM NUMBER ___

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

_____ , _____ , _____ , _____ , _____ , _____
Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f'(x) = \frac{f(x + h) - f(x - h)}{2h}$$

for all $x \in \mathbb{R}$ and all $h \neq 0$. (Hint: multiply both sides by $2h$.)
Suppose $f : [-1, 1] \to \mathbb{C}$ is a continuous complex-valued function, and for all non-negative integers $n$

$$\int_{-1}^{1} x^n f(x) dx = 0.$$ 

Prove that $f = 0.$
Problem 3B.  

The error of a quadrature rule with \( p + 1 \) distinct points \( x_j \), weights \( w_j \) is

\[
E_p(f) = \int_a^b f(x)dx - \sum_{j=0}^{p} w_j f(x_j).
\]

Suppose that \( E_p(f) = 0 \) whenever \( f \) is a polynomial of degree \( \leq q \). Show that \( q \leq 2p + 1 \) and if \( q \geq 2p \) then \( w_j > 0 \) for all \( j \).
Problem 4B.

Given \( n \) distinct points \( z_j \in \mathbb{C} \) and \( n \) values \( f_j \in \mathbb{C} \), show that there is a unique polynomial \( P \) of degree at most \( n - 1 \) such that

\[
P(z_j) = f_j
\]

for \( 1 \leq j \leq n \).
Write all values of $i^i$ in the form $a + bi$. 

Problem 5B. 

Score:
Problem 6B. 

Let $D$ be the unit disk in the complex plane $\mathbb{C}$, $f : D \to \mathbb{C}$ an analytic function with 

$$|f^{(k)}(0)| \leq M$$

for all $k \geq 0$, and let $t_p \in D$, $s_p \in D$ for $1 \leq p \leq n$. For each $n \geq 1$ define $A_{ij} = f(t_i s_j)$ for $1 \leq i, j \leq n$. For each $r \geq 1$ find an $n \times n$ matrix $B$ with rank $\leq r$ and 

$$|A_{ij} - B_{ij}| \leq \frac{2M}{r!}$$

for $1 \leq i, j \leq n$. 

Score:
Suppose $R$ is an invertible upper triangular complex matrix and $A$ is symmetric. Find an explicit formula for the entries of the upper triangular matrix $E$ satisfying

$$E^T R + R^T E = A$$

and show that your solution is unique. Hint: Multiply by $R^{-1T}$ and $R^{-1}$. 
Problem 8B.

Find a product of cyclic groups of prime power order isomorphic to \((\mathbb{Z}/1000000\mathbb{Z})^*\) (the group of units of the ring of integers mod 1000000).
Problem 9B.

Let $S_9$ denote the group of permutations of 9 objects.

(a) Exhibit an element of $S_9$ of order 20.

(b) Prove that no element of $S_9$ has order 18.