Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A — Spring Semester 2016

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, calculators or electronic devices may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Show that

$$\int_4^9 \sqrt{-6+5\sqrt{-6+5\sqrt{-6+5\sqrt{-6+5\sqrt{x}}}}}\,dx$$

is a rational number.

Solution:

YOUR EXAM NUMBER ____

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Problem 2A. *Score:*

Suppose that f and g are continuously differentiable real-valued functions on $\mathbb{R}$ with $f, g, f', g' \in L^2(\mathbb{R})$. Show that

$$\int_{-\infty}^{\infty} f g' \, dx = - \int_{-\infty}^{\infty} f' g \, dx.$$

(Recall that $L^2(\mathbb{R})$ is the set of integrable functions h such that $\int_{-\infty}^{\infty} |h|^2 \, dx < \infty$.)

Solution:

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Problem 3A. *Score:*

Suppose g and f_n are nonnegative integrable functions such that $\int f_n \, dx \to 0$ as $n \to \infty$ and $f_n^2 \leq g$ for all n. Prove or find a counterexample to the statement that $\int f_n^4 \, dx \to 0$ as $n \to \infty$.

Solution:

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Problem 4A. *Score:*

Prove that a monic polynomial $p(z)$ with real coefficients is real-rooted if and only if $\Im(p'(z)/p(z)) < 0$ whenever $\Im(z) > 0$. ($\Im(z)$ denotes the imaginary part of z.)

Solution:

YOUR EXAM NUMBER ____
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Problem 5A. *Score:*

Compute

$$\int_0^{2\pi} \frac{d\theta}{(3 + e^{-i\theta})^2}.$$

Solution:

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Problem 6A. *Score:*

Prove or disprove: there exists an $\epsilon > 0$ and a real matrix A such that

$$A^{100} = \begin{bmatrix} -1 & 0 \\ 0 & -1-\epsilon \end{bmatrix}.$$

Solution:

YOUR EXAM NUMBER ____

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Problem 7A. *Score:*

Suppose A is a symmetric matrix with rational entries and $A = UDU^T$, where U is orthogonal. Must D have rational entries? Prove or find a counterexample.

Solution:

YOUR EXAM NUMBER ___

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Problem 8A. *Score:*

Find a product of cyclic groups of prime power order isomorphic to the group of units in the ring of integers modulo 2016.

Solution:

Problem 9A. *Score:*

Compute the Galois group of the normal closure of the field

$$K = \mathbb{Q}(\sqrt{3} + \sqrt{5})$$

over $\mathbb{Q}$.

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2016

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

YOUR EXAM NUMBER ___

Please cross out this problem if you do not wish it graded

Problem 1B. *Score:*

Show that

$$\int_0^\infty \frac{t\,e^{-t/2}}{1-e^{-t}}\,dt = 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

Solution:

YOUR EXAM NUMBER ____
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Problem 2B. *Score:*

Let $(f_i)_{i=1}^{\infty}$ and g be twice-differentiable real-valued functions on $\mathbb{R}$, with $f_i'' \geq 0$. Suppose that

$$\lim_{i \to \infty} f_i(x) = g(x)$$

for all $x \in \mathbb{R}$. Show that $g'' \geq 0$.

Solution:

YOUR EXAM NUMBER ___

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Problem 3B. *Score:*

Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|}$$

converges pointwise to a Lipschitz function $f(x)$. Is the convergence uniform on $\mathbb{R}$?

Solution:

YOUR EXAM NUMBER ____
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Problem 4B. *Score:*

Compute

$$\int_C \frac{6z^5 + 1}{z^6 + z + 1}\, dz,$$

where C is the circle centered at the origin with radius 2, oriented counterclockwise.

Solution:

YOUR EXAM NUMBER ____
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Problem 5B. *Score:*

Let $f(z) = \sum f_n z^n$ and $g(z) = \sum g_n z^n$ define holomorphic functions on a neighborhood of the closed unit disk $D = \{z : |z| \leq 1\}$. Prove that $h(z) = \sum f_n g_n z^n$ also defines a holomorphic function on a neighborhood of D.

Solution:

Problem 6B. *Score:*

Let A be an $m \times n$ real matrix and $y \in \mathbb{R}^m$. Let $x \in \mathbb{R}^n$ be a vector with nonnegative entries that minimizes the Euclidean distance $\|y - Ax\|$ (among all nonnegative vectors x). Show that the vector $v = A^T(y - Ax)$ has nonnegative entries.

Solution:

YOUR EXAM NUMBER ____

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Problem 7B. *Score:*

Let A be a real square matrix and let ρ be the maximum of the absolute values of its eigenvalues (*i.e.*, its spectral radius). (1) Show that if A is symmetric then $\|Ax\| \leq \rho\|x\|$ for all $x \in \mathbb{R}^n$, where $\|\cdot\|$ denotes the Euclidean norm. (2) Is this true when A is not symmetric? Prove or give a counterexample.

Solution:

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Problem 8B. *Score:*

Factor the polynomial

$$f(x) = 6x^5 + 3x^4 - 9x^3 + 15x^2 - 13x - 2$$

into a product of irreducible polynomials in the ring $\mathbb{Q}[x]$.

Solution:

YOUR EXAM NUMBER ___
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Problem 9B. *Score:*

Let p be a prime number. Prove that every group G of order p^2 is commutative.

Solution: