Department of Mathematics, University of California, Berkeley

YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2015

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1. Please write your 1- or 2-digit exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

YOUR EXAM NUMBER ____

Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

(a) Evaluate the integral

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$$

(b) Prove that $0 < \frac{22}{7} - \pi < \frac{1}{256}$

Solution:

YOUR EXAM NUMBER ____

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Problem 2A. *Score:*

Suppose that g is a (not necessarily continuous) positive real valued function of a real number. If $a < b$ are real numbers, show that there is a finite sequence $a = t_0 < t_1 < \cdots < t_n = b$ of real numbers such that in each interval $[t_k, t_{k+1}]$ there is a point where the value of the function g is greater than the length of the interval.

Solution:

YOUR EXAM NUMBER ___

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Problem 3A. *Score:*

(a) Describe all sets of reals that can be the image of the real line under a polynomial with real coefficients.

(b) Find the image of the real plane under the polynomial $x^2 + (xy - 1)^2$.

(c) Describe all sets of reals that can be the image of the real plane under a polynomial in 2 variables with real coefficients.

Solution:

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Problem 4A. *Score:*

Write two different Laurent series in powers of the complex variable z for the function

$$f(z) = \frac{1}{z(1+z^2)} \ .$$

Give the domain of each of these series.

Solution:

YOUR EXAM NUMBER ___

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Problem 5A. *Score:*

Compute the difference

$$\int_{|z|=3} \frac{e^{\pi/z}\,dz}{z^2+4} - \int_{|z|=1} \frac{e^{\pi/z}\,dz}{z^2+4}\ ,$$

where both integrals are taken in the *counter-clockwise* direction.

Solution:

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Problem 6A. *Score:*

Fix $N \geq 1$. Let $s = (s_1, \ldots, s_N)$ and $t = (t_1, \ldots, t_N)$ be $2N$ distinct complex numbers. Define the $N \times N$ matrices $C(t,s)$, $P(t,s)$ and $Q(s)$ with P and Q diagonal to have entries

$$C(t,s)_{ij} = \frac{1}{t_i - s_j}, \qquad P(t,s)_{ii} = \prod_{k=1}^{N}(t_i - s_k), \qquad Q(s)_{jj} = \prod_{k \neq j} \frac{1}{s_j - s_k}$$

Show that $p(t) = P(t,s)C(t,s)Q(s)p(s)$, where p is any polynomial of degree less than N, and for a vector $r = (r_1, \ldots, r_N)$, $p(r)$ is defined to be the vector $(p(r_1), ..., p(r_N))$.

Solution:

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Problem 7A. *Score:*

Compute the determinant $\Delta_n = \begin{vmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{n-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{n}{1} \\ \binom{2}{2} & \binom{3}{2} & \binom{4}{2} & \cdots & \binom{n+1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n-1}{n-1} & \binom{n}{n-1} & \binom{n+1}{n-1} & \cdots & \binom{2n-2}{n-1} \end{vmatrix}$.

Solution:

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Problem 8A. *Score:*

Factor the polynomial

$$11x^5 - 11x^4 + 14x^2 - 21x + 7$$

into irreducible polynomials in $\mathbb{Q}[x]$.

Solution:

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Problem 9A. *Score:*

Find (with proof) a product of cyclic groups that is isomorphic to the group

$$(\mathbb{Z}_{12} \times \mathbb{Z}_{12})/\langle(2,6)\rangle$$

(Here $\mathbb{Z}_n$ means $\mathbb{Z}/n\mathbb{Z}$.)

Solution:

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YOUR 1 OR 2 DIGIT EXAM NUMBER ____

GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2015

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

YOUR EXAM NUMBER ____

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Problem 1B. *Score:*

For all integers $n > 2$ prove the inequality

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$$

Solution:

YOUR EXAM NUMBER ____
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Problem 2B. *Score:*

Find the maximum area of all triangles that can be inscribed in an ellipse with semiaxes a and b.

Solution:

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Problem 3B. *Score:*

Suppose that $a_{1,1} + a_{1,2} + \cdots,\qquad a_{2,1} + a_{2,2} + \cdots$, are a countable collection of convergent series of non-negative real numbers. Show that there is a convergent series $x_1 + x_2 + \cdots$ of real numbers converging more slowly than any of the given series in the sense that for any m we have $x_n \geq a_{m,n}$ for all sufficiently large n. (Hint: The problem is not affected by changing a finite number of terms of each of the given series.)

Solution:

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Problem 4B. *Score:*

Prove that there is no one-to-one conformal map from the punctured unit disk $\{z : 0 < |z| < 1\}$ onto the annulus $\{z : 1 < |z| < 2\}$.

Solution:

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Problem 5B. *Score:*

Show that if $f : \mathbb{C} \to \mathbb{C} \cup \infty$ is a meromorphic function in the plane, such that there exists $R, C > 0$ so that for $|z| > R$, $|f(z)| \leq C|z|^n$, then f is a rational function.

Solution:

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Problem 6B. *Score:*

What is the maximal dimension of subspaces in $\mathbb{R}^4$ on which the quadratic form $x_1x_2 - 3x_2^2 + x_3^2 + 2x_2x_4 + x_4^2$ is positive definite?

Solution:

YOUR EXAM NUMBER ____

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Problem 7B. *Score:*

Fix $N \geq 1$. Let $s_1, \ldots, s_N, t_1, \ldots, t_N$ be $2N$ complex numbers of magnitude less than or equal to 1. Let A be the $N \times N$ matrix with entries

$$A_{ij} = \exp\left(t_i s_j\right).$$

Show that A can be approximated by matrices of small rank in the following sense: for any $m \geq 1$ the $N \times N$ matrix B with entries $\sum_{n=0}^{m-1} \frac{(t_i s_j)^n}{n!}$ satisfies

$$|A_{ij} - B_{ij}| \leq \frac{2}{m!}$$

for all i and j and has rank at most m.

Solution:

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Problem 8B. *Score:*

Let p be a prime number and G be a group such that $g^p = 1$ for all $g \in G$. Show that if p=2 then G is abelian, and give an example with $p > 2$ where G is not abelian.

Solution:

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Problem 9B. *Score:*

Let p be a prime. Let $p^{a(n)}$ be the largest power of p dividing $n!$ and let $b(n)$ be the sum of the digits of n in base p.
(a) Show that $a(n) = [n/p] + [n/p^2] + [n/p^3] + \cdots$ where $[x]$ is the largest integer at most equal to x.
(b) Express $a(n)$ in terms of the digits d_k of the base p expansion $n = \sum d_k p^k$ of n (where $0 \le d_k < p$).
(c) Find a nontrivial linear relation between the functions n, $a(n)$ and $b(n)$ (with coefficients that may depend on p but do not depend on n).

Solution: