Department of Mathematics, University of California, Berkeley

STUDENT EXAM NUMBER ____________

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2014

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1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem p on either side of the page for problem q if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

GRADE COMPUTATION

1A. ______	1B. ______	Calculus
2A. ______	2B. ______	Real analysis
3A. ______	3B. ______	Real analysis
4A. ______	4B. ______	Complex analysis
5A. ______	5B. ______	Complex analysis
6A. ______	6B. ______	Linear algebra
7A. ______	7B. ______	Linear algebra
8A. ______	8B. ______	Abstract algebra
9A. ______	9B. ______	Abstract algebra

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

STUDENT EXAM NUMBER ________________________
Please cross out this problem if you do not wish it graded

Problem 1A. *Score:*

Find the sum of the series $\frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \cdots$.

Solution:

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Problem 2A. *Score:*

Prove or disprove that there is a sequence $\{f_n\}$ of continuous functions on $\mathbb{R}$ such that for any rational x the sequence $f_n(x)$ is bounded but the sequence $f_n(x+\sqrt{2})$ is unbounded.

Solution:

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Problem 3A. *Score:*

Find a real number c so that

$$\left| c - \int_{-1/2}^{1/2} \frac{\exp(x) - 1}{x} dx \right| < 0.01$$

.

Solution:

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Problem 4A. *Score:*

Let f be analytic on the closed unit disk, and assume that $|f(z)| \leq 1$ for all z's in this set. Suppose also that $f(\frac{1}{2}) = f(\frac{i}{2}) = 0$. Prove that $|f(0)| \leq \frac{1}{4}$.

Solution:

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Problem 5A. *Score:*

Let f, g be meromorphic functions on $\mathcal{C}$ such that $|f(z)| \leq |g(z)|$ at all z where both are defined. Show there is a $c \in \mathcal{C}$ such that $f(z) = cg(z)$ for all z where both are defined.

Solution:

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Problem 6A. *Score:*

Let R be a finite ring (with 1) of characteristic p. For S a subring of R (not necessarily containing an identity element), S is a vector space over F_p. For $a \in S$ let $T_a^S : S \to S$ be the linear map $T_a^S(x) = ax$.

(a) Show: if $1 \in S$ then the minimal polynomial of $T_a^S =$ the minimal polynomial of T_a^R.

(b) Give an example of p, R, S, a where (a) is false.

Solution:

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Problem 7A. *Score:*

Let F be a finite field with q elements. A *complete flag* in the vector space F^n is a nested sequence of linear subspaces $V^1 \subset V^2 \subset \cdots \subset V^{n-1}$ of dimensions $1, 2, \ldots, n-1$ respectively. Let $f_n(q)$ be the number of complete flags in F^n as a function of q. Find the limit of $f_n(q)$ as q tends to 1.

Solution:

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Problem 8A. *Score:*

Let G be a group of order 48. Show that G contains a normal subgroup of order 16 or 8.

Solution:

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Problem 9A. *Score:*

Let A be a finite abelian group (under $+$) and let $R = End(A)$ be the ring of homomorphisms from A to A. Show there is a subring S of R such that A and S are isomorphic as abelian groups.

Solution:

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GRADUATE PRELIMINARY EXAMINATION, Part B — Spring Semester 2014

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

STUDENT EXAM NUMBER ________________________

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Problem 1B. *Score:*

Let the continuously differentiable function $f : [0, 1]^2 \to \mathbb{R}$ on the unit square be given by the distance to a fixed point outside the square. Show that there there is no point (x_0, y_0) in the square such that the gradient of the function f at this point is equal to the average value of the gradient of the function f. In other words the obvious analogue of the mean value theorem for functions of two variables is false. (Hint:first find the length of the gradient of f.)

Solution:

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Problem 2B. *Score:*

Prove that a non-empty closed convex subset of the real vector space R^n with the usual Euclidean distance has a unique element of minimum norm (distance to the origin).

Solution:

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Problem 3B. *Score:*

Prove that there exists a constant C such that for every polynomial P of degree 2014

$$P(0) \leq C \int_0^1 |P(x)| \, dx.$$

Solution:

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Problem 4B. *Score:*

If f is an injective holomorphic function defined on the open unit disk U of the complex plane, show that the area of the image of U under f is $\int_U |f'|^2 dxdy$. Compute the area of the image of the unit disk U under the map $f(z) = z + (z^2)/2$.

Solution:

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Problem 5B. *Score:*

Find the integral $\int_0^\infty \frac{\cos x}{1+x^2} dx$.

Solution:

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Problem 6B. *Score:*

Let n be an integer and let $O(n)$ be the group of $n \times n$ orthogonal matrices. View $O(n)$ as a topological group with the induced topology from the embedding $O(n) \subset \mathbb{R}^{n^2}$ given by the entries. Show that $O(n)$ is compact.

Solution:

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Problem 7B. *Score:*

Let A be the matrix

$$A = \begin{pmatrix} 5/2 & 0 & -1/2 \\ 0 & 3 & 0 \\ 5/2 & 0 & -1/2 \end{pmatrix}.$$

Calculate A^{16}. (You may give your answer as a polynomial in A of degree at most 2.)

Solution:

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Problem 8B. *Score:*

Let $B = C^{-1}AC$, where A and C are $n \times n$-matrices with integer entries, such that $\det A = 1$, and $\det C \neq 0$. Prove that there exists a positive integer m such that all entries of B^m are integers.

Solution:

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Problem 9B. *Score:*

Let G be a finite group acting on a finite set X with a single orbit. For an element $g \in G$ let $\mathrm{Fix}_g(X)$ denote the set $\{x \in X | g(x) = x\}$.

(a) Show that

$$\#G = \sum_{g \in G} \#\mathrm{Fix}_g(X).$$

Hint: Count the set $S = \{(x, g) \in X \times G | gx = x\}$ two ways.

(b) Show that if X has more than 1 point then there exists an element $g \in G$ fixing no points of X.

Solution: