Department of Mathematics, University of California, Berkeley

GRADUATE PRELIMINARY EXAMINATION, Part A Spring Semester 2013

Problem 1A. *Score:*

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. Calculate the limit

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) \frac{\epsilon}{\epsilon^2 + t^2} dt$$

Solution: Multiplying t by ϵ gives the limit as

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(\epsilon t) \frac{1}{1 + t^2} dt$$

which as ϵ tends to 0 becomes

$$f(0) \int_{-\infty}^{\infty} \frac{1}{1 + t^2} dt = \pi f(0)$$

Problem 2A. *Score:*

Suppose that f is a smooth real function defined for all real x, such that $|f'(x)| \geq \epsilon > 0$ and $|f''(x)| \leq M > 0$ for all x.

(1) Show that f has a unique zero z.

(2) Given x_0, define a sequence by $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Show that

$$|x_{n+1} - z| \leq |x_n - z|^2 M/\epsilon.$$

(Hint: $f(x_n) = \int_z^{x_n} f'(x)dx$.)

(3) Show that the sequence $\{x_n\}$ converges to the zero z of f provided that $|f(x_0)| < \epsilon^2/M$.

Solution: Part 1 follows from the intermdiate value theorem and Rolles theorem in the usual way.

For part 2 we can assume $z = 0$. Then

$$x_n f'(x_n) - f(x_n) = \int_0^{x_n} (f'(x_n) - f'(x))dx$$

which has absolute value at most $|x_n| \times |x_n| M \leq x_n^2 M$ so

$$|x_{n+1}| \leq |x_n|^2 M/\epsilon.$$

For part 3, note that part 2 shows that each term of the sequence $|x_n - z|M/\epsilon$ is bounded by the square of the previous term, so the sequence tends to zero if the first term is less than 1, which follows from $|f(x_0)| < \epsilon^2/M$.

Problem 3A. *Score:*

Show that $\int_0^\infty x \exp(-x^6(\sin x)^2)dx$ is finite.

Solution: More generally, the convergence of the integral $\int_0^\infty x^\alpha \exp(-x^\beta(\sin x)^2)dx$ depends on the size of the "spikes" at $x = n\pi$ where $\sin(x) = 0$. The size of the spike at $n\pi$ is bounded by a constant times

$$n^\alpha \int_{-\infty}^{\infty} \exp(-n^\beta x^2)dx$$

which is bounded by a constant times

$$n^{\alpha-\beta/2}.$$

So the integral converges if the series $\Sigma n^{\alpha-\beta/2}$ does, which is true if $\alpha - \beta/2 < -1$, in particular if $\alpha = 1$, $\beta = 6$.

Problem 4A. *Score:*

Find

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)}\,dz$$

when C is the circle $|z| = 2$, described in the positive sense.

Solution: The singularities inside of C are at $z = 0$, i and $-i$. The residue at 0 is 1. The residues at i and $-i$ are $\frac{1}{2}$. The integral is $4\pi i$.

Problem 5A. *Score:*

Let $n \geq 1$ and let $\{a_0, a_1, \ldots, a_n\}$ be complex numbers such that $a_n \neq 0$. For $\theta \in \mathbf{R}$, define

$$f(\theta) = a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + \ldots + a_n e^{ni\theta}.$$

Prove that there exists $\theta \in \mathbf{R}$ such that $|f(\theta)| > |a_0|$.

Solution: Suppose that the claim is false. Then for all $\theta \in \mathbf{R}$, we have $|f(\theta)| \leq |a_0|$. Put $g(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ so that $f(\theta) = g(e^{i\theta})$. By the maximum modulus principle, applied to $K = \{z : |z| \leq 1\}$, we know that $|g|$ is maximized on some boundary point of K. Hence $|g|$ also has an interior maximum at 0, so g must be a constant function. This contradicts the assumptions that $n \geq 1$ and $a_n \neq 0$.

Problem 6A. *Score:*

Show that if V is a real vector space with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$ and $W \subset V$ is a linear subspace then $W^\perp = ((W^\perp)^\perp)^\perp$. Give an example such that $W \neq (W^\perp)^\perp$.

Solution: For any subspace we have $X \subset X^{\perp\perp}$ as X is orthogonal to $X^\perp$, so applying this to $X = W^\perp$ shows that $W^\perp \subset ((W^\perp)^\perp)^\perp$. On the other hand, If $X \subset Y$ then $Y^\perp \subset X^\perp$. Applying this to $X = W$, $Y = W^{\perp\perp}$ shows that $((W^\perp)^\perp)^\perp \subset W^\perp$. So $((W^\perp)^\perp)^\perp = W^\perp$.

For the example, take $V = \ell^2(\mathbb{N})$ and $W \subset V$ the subspace of eventual ly 0 sequences. Then $W^\perp = \{0\}$ so $(W^\perp)^\perp = V$ but $(1, 1/2, 1/3, \ldots) \notin W$.

Problem 7A. *Score:*

Let A be a matrix over the field of complex numbers. Suppose A has finite order, in other words $A^m = I$ for some positive integer m. Prove that A is diagonalizable. Give an example of a matrix of finite order over an algebraically closed field that is not diagonalizable.

Solution: By a standard theorem of linear algebra, A is diagonalizable if and only if its minimal polynomial has no repeated roots. The hypothesis implies that the minimal polynomial of A divides $X^m - 1$. Hence it has distinct roots, since $X^m - 1$ does.

The matrix $\binom{11}{01}$ over an algebraically closed field of characteristic $p > 0$ has finite order p but is not diagonalizable: both eigenvalues are 1, so if it were diagonalizable it would have to be the identity matrix.

Problem 8A. *Score:*

Let m and n be integers greater than 1. Prove that $\log_m(n)$ is rational if and only if $m = l^r$ and $n = l^s$, for some positive integers l, r, and s.

Solution: If $m = l^r$ and $n = l^s$, then $\log_m(n) = s/r$. Conversely, suppose $\log_m(n) = s/r$, where s and r are integers which we may assume coprime and positive ($n > 1$ implies

$\log_m(n) > 0$). Then $m^s = n^r$. By the fundamental theorem of arithmetic, the prime factorizations of m and n must be of the form $m = p_1^{e_1} \cdots p_k^{e_k}$, $n = p_1^{f_1} \cdots p_k^{f_k}$, where $se_i = rf_i$ for all i. Since r and s are coprime, this implies $e_i = rh_i$, $f_i = sh_i$ for some h_i. Hence $m = l^r$ and $n = l^s$, where $l = p_1^{h_1} \cdots p_k^{h_k}$.

Problem 9A. *Score:*

Let K be a field. Let R be an integral domain which contains K and is finite-dimensional (as a vector space) over K. Prove that R is a field.

Solution: Let $x \in R$, $x \neq 0$. The map $m_x \colon R \to R$ defined by $m_x(r) = xr$ is a linear endomorphism of R as a vector space over K. Since R is an integral domain, the kernel of m_x is zero, *i.e.*, m_x is injective. Since R is finite-dimensional, this implies that m_x is surjective. Then the element y such that $m_x(y) = 1$ is an inverse of x.

Alternate solution: the fact that R is an integral domain implies that the minimal polynomial $P(X)$ of x over K is irreducible. In particular, its constant term c is non-zero. The identity $P(x) = 0$ can be rewritten as $xQ(x) = -c$, where $P(X) - c = XQ(X)$, so $y = -Q(x)/c$ is an inverse of x.

Problem 1B. *Score:*

Find $\int_0^1 \arctan(x)dx$.

Solution: Integration by parts gives

$$1 \times \arctan(1) - \int_0^1 \frac{x}{1+x^2}dx = \pi/4 - (\log 2)/2.$$

Problem 2B. *Score:*

Prove that the intersection of a decreasing sequence of closed connected subsets of a compact metric space is connected. Give an example to show that this is false if the assumption that the space is compact is dropped.

Solution:

Suppose that $X_1, X_2, ...$ is a decreasing sequence of closed connected subsets with intersection X. If X is not connected, it is the union of non-empty disjoint closed subsets Y and Z. Pick disjoint open subsets U and V containing Y and Z. Then U, V, and the complements of the X_i form an open cover of a compact space, which therefore has a finite subcover, so some X_i is contained in the union of U and V. But this contradicts the fact that X_i is connected.

The sequence of connected subsets of the plane consisting of the union of the set $x = 0$, $x = 1$, $y \geq n$ has disconnected intersection.

Problem 3B. *Score:*

Let g be 2π-periodic, continuous on $[-\pi, \pi]$ and have Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,.$$

Let f be 2π-periodic and satisfy the differential equation

$$f''(x) + kf(x) = g(x)$$

where $k \neq n^2, n = 1, 2, 3, \ldots$. Find the Fourier series of f and prove that it converges everywhere.

Solution:

$$f(x) = \frac{a_0}{2k} + \sum_{n=1}^{\infty}(\frac{a_n}{k-n^2}\cos nx + \frac{b_n}{k-n^2}\sin nx)\,.$$

This converges (uniformly) for all x as the numbers a_n and b_n are bounded, and the series $\sum \frac{1}{k-n^2}$ converges.

Problem 4B. *Score:*

Let U be an open subset of $\mathbf{C}$. Let K be a closed bounded subset of $\mathbf{C}$ that is contained in U. Put

$$D = \min_{p \in K, q \notin U} |p - q|.$$

That is, D is the closest distance between K and $\mathbf{C} - U$. (If $U = \mathbf{C}$ then we put $D = \infty$.)

Suppose that f is an analytic function on U so that for all $z \in U$, we have $|f(z)| \leq M$. Here M is a fixed positive number. Find an explicit number $C < \infty$, depending on M and D, so that for all $z_0 \in K$ we have $|f'(z_0)| \leq C$. Justify your answer.

Solution: Given $z_0 \in K$ and $r < D$, put $C_r = \{z : |z - z_0| = r\}$. Then f is analytic in and on C_r. We have

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^2}\, dz,$$

so $|f'(z_0)| \leq \frac{1}{2\pi} \cdot \frac{M}{r^2} \cdot 2\pi r = \frac{M}{r}$. Taking $r \to D$, we can put $C = \frac{M}{D}$.

Problem 5B. *Score:*

Which of the following domains are biholomorphically equivalent to each other: the complex plane $\mathbb{C}$, the unit disk $D \subset \mathbb{C}$, the upper halfplane $\mathbb{H} \subset \mathbb{C}$? Write explicit biholomorphisms or prove they cannot exist.

Solution: An explicit biholomorphism $\phi : \mathbb{H} \to D$ is given by

$$\phi(z) = \frac{z - i}{z + i}$$

If a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is bounded then it is constant. In particular, any holomorphic function $f : \mathbb{C} \to D \subset \mathbb{C}$ is constant and hence not a biholomorphism.

Problem 6B. *Score:*

Show that the $n \times n$ (Cauchy) matrix with entries $1/(x_i - y_j)$ has determinant

$$\frac{\prod_{1 \leq j < i \leq n} (x_i - x_j)(y_j - y_i)}{\prod_{1 \leq i,j \leq n} (x_i - y_j)}$$

Solution: Multiplying the determinant by $\prod_{1 \leq i,j \leq n} (x_i - y_j)$ gives a polynomial of degree $n(n-1)$. This polynomial vanishes whenever two x's or two y's are equal so is divisible by $\prod_{1 \leq j < i \leq n} (x_i - x_j)(y_j - y_i)$, and therefore equal to a constant times this as the degrees are the same. The constant can be checked to be 1 by looking at the coefficient of some monomial.

Problem 7B. *Score:*

Prove the following three statements about real $n \times n$ matrices.
1. If A is an orthogonal matrix whose eigenvalues are all different from -1, then $I + A$ is nonsingular and $S = (I - A)(I + A)^{-1}$ is skew-symmetric.

2. If S is a skew-symmetric matrix, then $A = (I - S)(I + S)^{-1}$ is an orthogonal matrix with no eigenvalue equal to -1.
3. The correspondence (called the Cayley transform) $A \leftrightarrow S$ from Parts 1 and 2 is one-to-one.

Solution: For part 1, $S = I + A$ has no eigenvalues 0 so is non-singular. Its transpose is $(I + A^T)^{-1}(I - A^T) = (I + A^{-1})^{-1}(I - A^{-1}) = (A + I)^{-1}(A - I) = -S$ so S is skew symmetric. Part 2 is similar to part 1 (noting that all eigenvalues of S are imaginary so $1 + S$ is invertible). Since the maps in parts 1 and 2 are inverses we get a 1:1 bijection.

Problem 8B. *Score:*

Consider the symmetric group Σ_n in its presentation as $n \times n$ permutation matrices. Define the "expected trace" to be the weighted sum of traces

$$E_n = \frac{1}{n!} \sum_{g \in \Sigma_n} \text{Trace}(g)$$

Calculate E_n.

Solution: The ith diagonal entry of the permutation matrix g is equal to 1 for exactly $(n-1)!$ elements g since such g can be regarded as elements of Σ_{n-1}. Thus summing over i, we find $E_n = n(n-1)!/n! = 1$.

Problem 9B. *Score:*

If F is a finite field, show that more than half the elements of F are squares. Show that every element is the sum of 2 squares.

Solution: Any non-zero element has at most 2 square roots, so at least half the non-zero elements are squares. The element 0 is also a square, so more than half the elements are squares.

If b is any element of the finite field, then the sets of elements of the form x^2 and $b - y^2$ both contain more than half the elements of the field, so they have an element in common. So $x^2 = b - y^2$ for some x and y, so b is the sum of the squares of x and y.