1. Please write your 1- or 2-digit student exam number on this cover sheet and on all problem sheets (even problems that you do not wish to be graded).

2. Indicate below which six problems you wish to have graded. Cross out solutions you may have begun for the problems that you have not selected.

3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem \( p \) on either side of the page for problem \( q \) if \( p \neq q \).

4. No notes, books, or calculators may be used during the exam.

PROBLEM SELECTION

Part A: List the six problems you have chosen:


GRADE COMPUTATION

1A. _____ 1B. _____  Calculus
2A. _____ 2B. _____  Real analysis
3A. _____ 3B. _____  Real analysis
4A. _____ 4B. _____  Complex analysis
5A. _____ 5B. _____  Complex analysis
6A. _____ 6B. _____  Linear algebra
7A. _____ 7B. _____  Linear algebra
8A. _____ 8B. _____  Abstract algebra
9A. _____ 9B. _____  Abstract algebra

Part A Subtotal: _____ Part B Subtotal: _____ Grand Total: _____
Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. Calculate the limit

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) \frac{\epsilon}{\epsilon^2 + t^2} dt$$

Solution:
Problem 2A.

Suppose that $f$ is a smooth real function defined for all real $x$, such that $|f'(x)| \geq \epsilon > 0$ and $|f''(x)| \leq M > 0$ for all $x$.

(1) Show that $f$ has a unique zero $z$.

(2) Given $x_0$, define a sequence by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. Show that

$$|x_{n+1} - z| \leq |x_n - z|^2 \frac{M}{\epsilon}.$$ 

(Hint: $f(x_n) = \int_z^{x_n} f'(x)dx$.)

(3) Show that the sequence $\{x_n\}$ converges to the zero $z$ of $f$ provided that $|f(x_0)| < \epsilon^2/M$.

Solution:
Problem 3A.  

Show that \( \int_{0}^{\infty} x \exp(-x^6 \sin^2 x) \, dx \) is finite.

Solution:
Find

\[ \int_C \frac{\cosh(\pi z)}{z(z^2 + 1)} \, dz \]

when \( C \) is the circle \( |z| = 2 \), described in the positive sense.

Solution:
Let $n \geq 1$ and let $\{a_0, a_1, \ldots, a_n\}$ be complex numbers such that $a_n \neq 0$. For $\theta \in \mathbb{R}$, define
$$f(\theta) = a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + \ldots + a_n e^{ni\theta}.$$
Prove that there exists $\theta \in \mathbb{R}$ such that $|f(\theta)| > |a_0|$.

Solution:
Problem 6A.

Show that if $V$ is a real vector space with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$ and $W \subseteq V$ is a linear subspace then $W^\perp = ((W^\perp)^\perp)^\perp$. Give an example such that $W \neq (W^\perp)^\perp$.

Solution:
Let $A$ be a matrix over the field of complex numbers. Suppose $A$ has finite order, in other words $A^m = I$ for some positive integer $m$. Prove that $A$ is diagonalizable. Give an example of a matrix of finite order over an algebraically closed field that is not diagonalizable.

Solution:
Problem 8A.

Let $m$ and $n$ be integers greater than 1. Prove that $\log_m(n)$ is rational if and only if $m = lr$ and $n = ls$, for some positive integers $l$, $r$, and $s$.

Solution:
Let $K$ be a field. Let $R$ be an integral domain which contains $K$ and is finite-dimensional (as a vector space) over $K$. Prove that $R$ is a field.

Solution:
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PROBLEM SELECTION

Part B: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______
Problem 1B.  

Find $\int_0^1 \arctan(x) \, dx$.

Solution:
Problem 2B.

Prove that the intersection of a decreasing sequence of closed connected subsets of a compact metric space is connected. Give an example to show that this is false if the assumption that the space is compact is dropped.

Solution:
Let $g$ be $2\pi$-periodic, continuous on $[-\pi, \pi]$ and have Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Let $f$ be $2\pi$-periodic and satisfy the differential equation

$$f''(x) + kf(x) = g(x)$$

where $k \neq n^2, n = 1, 2, 3, \ldots$. Find the Fourier series of $f$ and prove that it converges everywhere.

Solution:
Problem 4B.

Let $U$ be an open subset of $\mathbb{C}$. Let $K$ be a closed bounded subset of $\mathbb{C}$ that is contained in $U$. Put

$$D = \min_{p \in K, q \notin U} |p - q|.$$ 

That is, $D$ is the closest distance between $K$ and $\mathbb{C} - U$. (If $U = \mathbb{C}$ then we put $D = \infty$.)

Suppose that $f$ is an analytic function on $U$ so that for all $z \in U$, we have $|f(z)| \leq M$. Here $M$ is a fixed positive number. Find an explicit number $C < \infty$, depending on $M$ and $D$, so that for all $z_0 \in K$ we have $|f'(z_0)| \leq C$. Justify your answer.

Solution:
Which of the following domains are biholomorphically equivalent to each other: the complex plane $\mathbb{C}$, the unit disk $D \subset \mathbb{C}$, the upper halfplane $\mathbb{H} \subset \mathbb{C}$? Write explicit biholomorphisms or prove they cannot exist.

Solution:
Problem 6B.

Show that the \( n \times n \) (Cauchy) matrix with entries \( 1/(x_i - y_j) \) has determinant

\[
\prod_{1 \leq j < i \leq n} (x_i - x_j)(y_j - y_i) \\
\prod_{1 \leq i, j \leq n} (x_i - y_j)
\]

Solution:
Problem 7B. Prove the following three statements about real $n \times n$ matrices.

1. If $A$ is an orthogonal matrix whose eigenvalues are all different from $-1$, then $I + A$ is nonsingular and $S = (I - A)(I + A)^{-1}$ is skew-symmetric.
2. If $S$ is a skew-symmetric matrix, then $A = (I - S)(I + S)^{-1}$ is an orthogonal matrix with no eigenvalue equal to $-1$.
3. The correspondence (called the Cayley transform) $A \leftrightarrow S$ from Parts 1 and 2 is one-to-one.

Solution:
Consider the symmetric group $\Sigma_n$ in its presentation as $n \times n$ permutation matrices. Define the “expected trace” to be the weighted sum of traces

$$E_n = \frac{1}{n!} \sum_{g \in \Sigma_n} \text{Trace}(g)$$

Calculate $E_n$.

Solution:
If $F$ is a finite field, show that more than half the elements of $F$ are squares. Show that every element is the sum of 2 squares.

Solution: