George M. Bergman
959 Evans Hall

Spring 1996, Math 54, Lec. 2
Make-up Final Exam

20 May, 1996
12:50-3:50PM

1. (35 points) Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 8 & 3 & 4 \end{bmatrix}$.

(a) (12 points) Find the characteristic polynomial of A, the eigenvalues of A, and a basis for the eigenspace of each eigenvalue.

(b) (12 points) Find the general solution to the differential equation $\mathbf{x}' = A\mathbf{x}$.

(c) (7 points) Show that the system of two differential equations

$$\begin{aligned} u'' &= -u' - u - v, \\ v' &= 3u' + 8u + 4v, \end{aligned}$$

can be reduced to a system of three *first-order* differential equations, namely the system given in part (b) above; and use the result of (b) to obtain the general solution to the above equations in u and v. (Alternatively, you may use any method to solve the above system of two equations, if you show your work and get the right answer.)

(d) (4 points) Obtain the particular solution to this system that satisfies $u(0) = u'(0) = 0$, $v(0) = 9$.

2. (10 points) Let f and g be differentiable functions on an interval I.
(a) (3 points) Define the Wronskian, $W(f, g)$.

(b) (7 points) Prove that if there is some $x \in I$ such that $W(f, g)(x) \neq 0$, then f and g are *linearly independent* functions; or equivalently, that if f and g are linearly *dependent*, then $W(f, g)(x) = 0$ for all x.

3. (15 points) Suppose p is a continuous function on the real line, and we wish to study functions $u(x, y)$ satisfying the partial differential equation

$$u_{yy}(x, y) = p(x)\, u_x(x, y).$$

(a) (8 points) Obtain the conditions two functions $a(x)$, $b(y)$ must satisfy for the function $u(x, y) = a(x)\, b(y)$ to be a solution to the above partial differential equation, in the form of a differential equation to be satisfied by each function, involving a constant in common. (This will be simpler than the considerations in Boyce and DiPrima for the heat and wave equations, because we are not imposing boundary conditions.)

(b) (7 points) Find (using the result of (a) or any other method) some nonconstant solution to the partial differential equation $u_{yy}(x, y) = x\, u_x(x, y)$.

4. (10 points) Let V be the subspace of C^3 spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -i \end{bmatrix}$.

Find an orthonormal basis of V with respect to the standard complex inner product on C^3.

5. (12 points) Given that one solution to the differential equation

$$x^2 y'' - 2y = 0$$

is $y_1(x) = x^{-1}$, find a second solution $y_2(x)$, which is not a scalar multiple of $y_1(x)$.

6. (18 points) Let f be the function on the interval $[0, 1]$ such that $f(x)$ is 0 for $0 \le x \le ½$, and 1 for $½ < x \le 1$.

(a) (9 points) Find the coefficients b_n in the Fourier expansion for this f of the form

$$\Sigma_{n=1}^{\infty} b_n \sin \pi n x.$$

(You may express these coefficients using values of trigonometric functions, without evaluating these for each n.)

(b) (5 points) Using the above result, find the solution to the partial differential equation

$$u_t(x,t) = \alpha^2 u_{xx}(x,t) \qquad (0 < x < 1,\ 0 < t)$$

subject to the initial conditions $u(x,0) = 0$ for $0 < x < ½$ and $u(x,0) = 1$ for $½ < x < 1$, and the boundary conditions $u(0,t) = 0 = u(1,t)$ for $t > 0$.

(c) (4 points) Sketch the function to which the series you found in part (a) converges on the interval $[-2, +2]$, indicating with a heavy dot the value of this function at each point of discontinuity.